Games, Puzzles, and Computation

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1 Introduction

The Sums of Positions is the combination of positions from different games that from a new position that has an equivalent position in another game. By determining the type of position in the sum we find the type position in an equivalent game.

2 Sums of Position

- already used sums of component

Def'n: δ, α, β are positions in normal play games. Define $\alpha + \beta$ to be a new position consisting of components α and β. To move in $\alpha + \beta$, a player chooses a component to move in.

- A player moves from $\alpha + \beta$ to either $\alpha' + \beta$ or $\alpha + \beta'$.

In Cut Cake, Player L cuts vertical, and Player R cuts horizontal, and whichever player who cannot do their move loses the game.

In Pick Up Bricks, there are N amount of bricks, each player can choose one or two bricks, and the player to pick up the last brick loses.

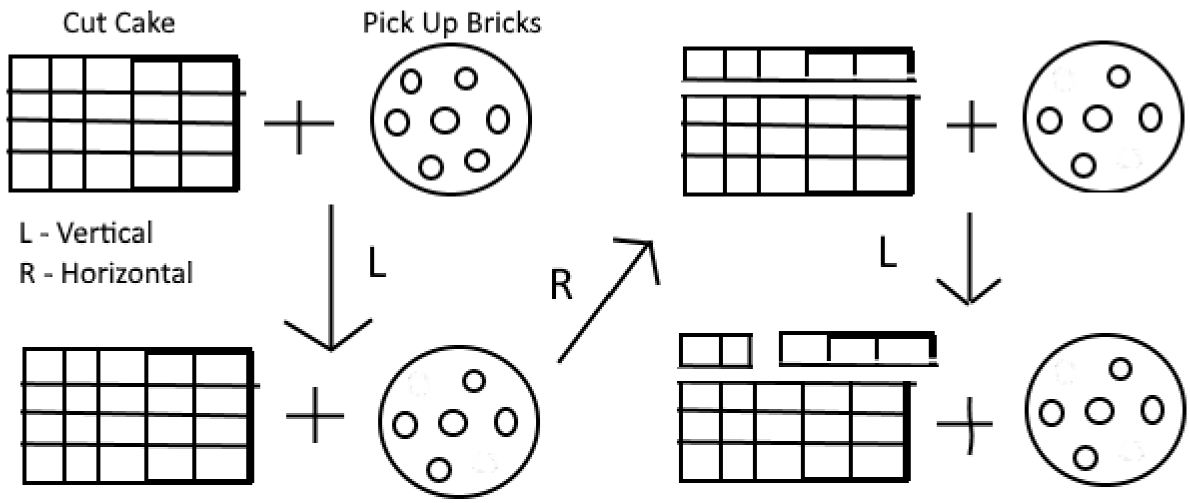
3 Determinate Sums

What is our strategy/ how do types behave under sums? Determine the type of the sum based on the type of the components.

Ex. Let α be some position in some game of type R
α + P$\cup$B

What is R's move?

- Since the P$\cup$B game is type P, it doesn't change anything.
- Ignore P$\cup$B game until L makes a move. Then respond in that component.



- Normal play games - last move wins, so next player plas in other comp.

Prop. If β is type P, then α and $\alpha + \beta$ are the same type.

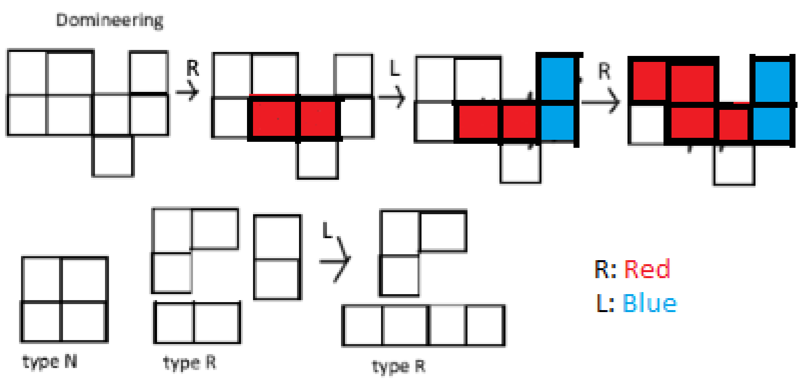
Prop. If α and β are both type L(R) then $\alpha + \beta$ is type L(R).

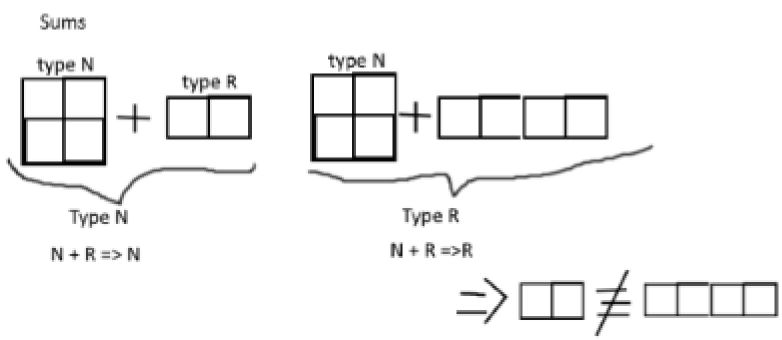
T	L	R	N	P
L	L	?	?	L
R	?	R	?	?
N	?	?	?	N
P	L	R	N	P

4 Intermediate Sums

Domineering - cover some grid with dominoes which cover 2 spots at once.

Assume R is horizontal, and L is vertical.



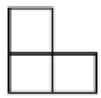


5 Equivalence

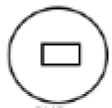
Move beyond one game at a time. Players can play two games, and see that there is equivalence between the two.

The next player to move will win either game.

Def'n: Two positions α and α' in (possibly different) normal play games are equivalent if for every position β in any normal play games, the two positions



Cut Cake



PUB

$\alpha + \beta$ and $\alpha' + \beta$ have the same type.

Equivalence Relations

Prop. If α, β, δ are positions in normal play games.

1. $\alpha \equiv \alpha$ (reflexivity)
2. $\alpha \equiv \beta \rightarrow \beta = \alpha$ (symmetry)
3. $\alpha \equiv \beta$ and $\beta \equiv \delta \rightarrow \alpha \equiv \delta$ (transitivity)

How does equivalence relate to type?

- If two positions are equivalent $\rightarrow$ they have the same type

Prop. If $\alpha \equiv \alpha'$, then they have the same type.

- Let β be a position in a normal play game with no moves left.
  $\alpha \leftrightarrow \alpha + \beta \leftrightarrow \alpha' + \beta \leftrightarrow \alpha'$
  - Algebra w/ +, $\equiv$
  - Prop. If α, β, δ are positions in normal play games.
    1. $\alpha + \beta \equiv \beta + \alpha$ (commutably)
    2. $(\alpha + \beta) + \delta \equiv \alpha + (\beta + \delta)$ (associatably)

Lemma. Given position α, β,δ in normal play games

1. If $\alpha \equiv \alpha'$, then $\alpha + \beta \equiv \alpha' + \beta$
2. If $\alpha_i \equiv \alpha_i$' for $1 \leq$ i $\leq$ n, then $\alpha_i + ... + \alpha_n \equiv \alpha_i$' + α_n'
3. If $\alpha_i \equiv \alpha_i$ for $1 \leq$ i $\leq$ m and $\beta_j \equiv \beta_j$' for $1 \leq$ j $\leq$ n, then $\{\alpha_n$,α_m — β_1,...,β_n $\} \equiv \{\alpha_1$,...,α_n' — β_1,...,β_n $\}$

Type P is equivalent to zero under normal addition.
Lemma. If β is type P, then $\alpha + \beta = \alpha$

Prop. If α and α' are type P, then $\alpha \equiv \alpha$' $\rightarrow \alpha + \delta \equiv \delta \equiv \alpha$' + δ

Lemma. If $\alpha + \beta$ and α' + β are both type P, then $\alpha \equiv \alpha$'

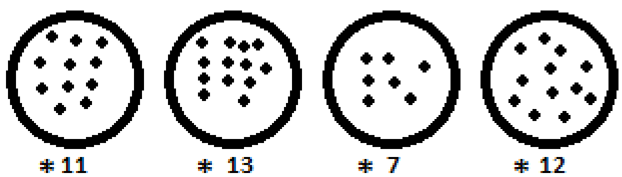
Cor. Every position in an impartial game is one of the following:

- Type N - The next player to play has a winning strategy.
- Type P - The previous (or second) player to play has a winning strategy.

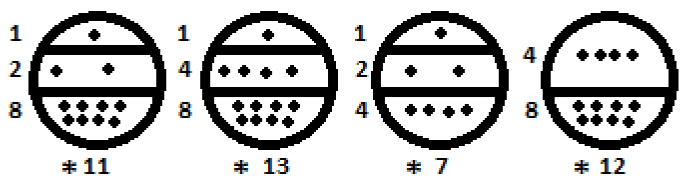
Cor. A position in an impartial game is:

- Type N - If $\exists$ a move to a position of type P.
- Type P - If there is no move to a position of type P.

For convenience, # of stones in each pile



Def'n: Given a Nim position $*\alpha_1 + *\alpha_2 + ... + *\alpha_k$, it is balanced if, for every power of 2, the total # of subpiles of that size is even.

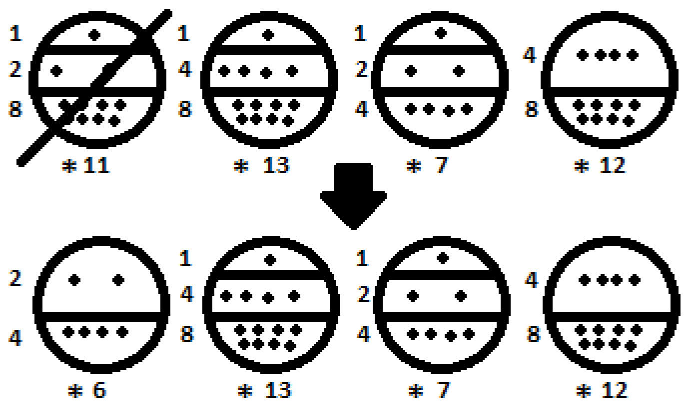


$2^0 + 2^1 + 2^2 + ... + 2^n < 2^{n+1}$

Procedure - Balance an unbalanced position:

- Let $*\alpha_1 + *\alpha_2 + ... + *\alpha_k$ be an unbalanced game of Nim.
- Suppose 2^n is the largest power of 2 which there are an odd # of subpiles Curretly all subpiles of $2^i, j > m$are balanced for every j ¡ m, if there are an odd # of subpiles of S^j, excluding this pile, leave s^i stones. Since at least 2^m stones could be removed, and $2^0 + 2^i + ... + 2^{n-1} < 2^m$ stones should be left, this is a legal move.

Prop. Every balanced Nim position is type P and every unbalanced Nim position is type N.



6 Nimbers

Its helpful to think of Nim positions like #'s

- we call a stack $*\alpha$ a nimber
- Based on prop and that P doesn't additively change anything a type P position *0
- $\alpha \equiv \gamma$
- every balanced position $\gamma \equiv *0$

Imagine the 2^{nd} player could add a stack to the game before it starts
Original $*a_1 + ... + *a_l$

- add *b
- to a balanced position of type P to determine odd # of subpiles of 2^j

$*a_1 + ... + *a_l + *b = *0$

7 Summary

When studying a position in a combinatorial game we may be unsure how to analyze it. We are instead better able to study this position by finding an equivalent position from the Sums of Positions of other games. When we analyze the Sums of Positions in simpler games we are able to find an equivalent position in an other game.