Games, Puzzles, & Computation

CSCI 4341 notes for 2/20/17 - 2/25/17

Group E

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Abstract

This week was broadly about Impartial games and Partisan games. In particularly, the Nim-sum, Nim positions, the MEX principle, Integer positions, Fraction Positions and the basics of the Dyadic numbers.

1 Announcements

1. Homework 4 is posted, due March 6.
2. Game for projects are updated.
3. New game reference papers are posted, there are some undergraduate student work included. It might be helpful for group project.
4. Preliminary notes will be posted on course website.

2 Notes

$\boldsymbol{*a_1, + * a_2 + ... + *a_l}$ (l Nim stacks)

Suppose the 2^{nd} player can add a stack, $*b$, at the beginning of the game.
Playing optimally, the 2^{nd} player must assure the tree is balanced after adding the $*b$ stack.

$\boldsymbol{\rightarrow *a_1, + * a_2 + ... + *a_l + *b \equiv *0}$ (adding $*b$ balances stacks resulting in a P position)

How do we determine the odd number of sub-piles of $\boldsymbol{2^j}$?

Definition: The *Nim-sum* of the non-negative integers $\boldsymbol{a_1, ..., a_l}$ denoted $\boldsymbol{a_1 \oplus a_2 \oplus ... \oplus a_l}$ is the non-negative integer $\boldsymbol{b}$ with the property that $\boldsymbol{2^j}$ appears in the binary expansion of $\boldsymbol{b}$ iff this term appears an odd number of times in the expansions of $\boldsymbol{a_1, ..., a_l}$

Example: The Nim-Sum of $13 \oplus 19 \oplus 10 \equiv (\not{8} + 4 + \not{1}) \oplus (16 + \not{2} + \not{1}) \oplus (\not{8} + \not{2}) \equiv 16 + 4 = 20$

- Break down number to powers of 2
- Cross-out the paired numbers
- Remaining values are the Nim-Sum

Nim-sum calculation in binary:

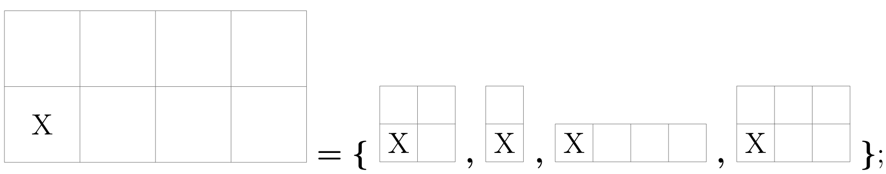
01101_2
10011_2
$\underline{01010_2}$
$10100_2 = 20_{10}$ (twenty in decimal)

Theorem: if $a_1, ..., a_l$ are non-negative and $b \equiv a_1 \oplus a_2 \oplus ... \oplus a_l$, then $*a_1 + *a_2 + ... + *a_l \equiv *b$

Proof Sketch:
$\boldsymbol{*a_1, + * a_2 + ... + *a_l + *b \equiv *0}$ (eq. 1)
$\boldsymbol{*b + *b \equiv *0}$ (since it's a copy, balance is maintained)
$\boldsymbol{*a_1 + ... + *a_l \equiv *a_1, +... + *a_l + *0}$ (adding balanced nimbers does not change type)
$\boldsymbol{\equiv *a_1 + ... + *a_l + *b + *b}$
$\boldsymbol{\equiv *0 + *b}$ (via eq.1)
$\boldsymbol{\equiv *b}$

- Every Nim position is equivalent to a nimber.
- For impartial games $\boldsymbol{\alpha = \{\beta_1, ..., \beta_l | \gamma_1, ..., \gamma_m\}}$ L, R have the same moves $\boldsymbol{\alpha = \{\alpha_1, ..., \alpha_k\}}$

Example: Nim $\boldsymbol{*4 \equiv \{*0, *1, *2, *3\}}$ (a stack of 4 nimbers)
Chop



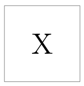
Definition: For a set $\boldsymbol{S = \{a_1, a_2, ..., a_n\}}$ of non-negative integers, the Minimum Excluded (MEX) value of S is the smallest non-negative integer b, which is not one of $\boldsymbol{a_1, ...a_n}$

Examples:
{0, 1, 2, 4, 7} MEX = 3
{1, 2, 8, 12} MEX = 0

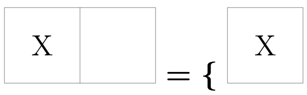
Theorem: (The MEX Principle) Let $\boldsymbol{\alpha = \{\alpha_1, ..., \alpha_k\}}$ be a position in an impartial game
Suppose $\boldsymbol{\alpha_i \equiv *a_i}$ for every $\boldsymbol{1 \leq i \leq k}$ (one α to one a)
Then $\boldsymbol{\alpha \equiv *b}$ where b is the MEX of the set $\{\alpha_1, ..., \alpha_k\}$

Theorem: (Sprague - Grundy) Every position in an impartial game is equivalent to a nimber

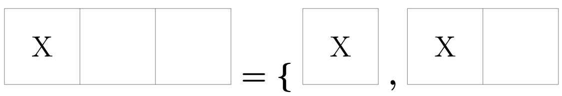
- Nim, PUB (Pick-Up Bricks), Chop, Chomp
- Chop



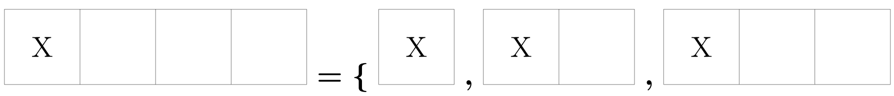
$\equiv \mathbf{*0}$



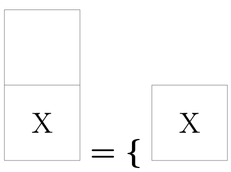
$\mathbf{\} \equiv \{*0\} \equiv *1}$



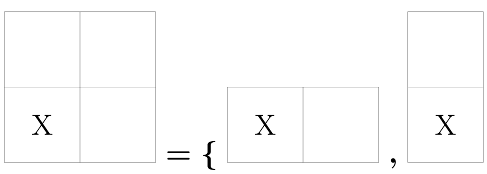
$\mathbf{\} \equiv \{*0, *1\} \equiv *2}$



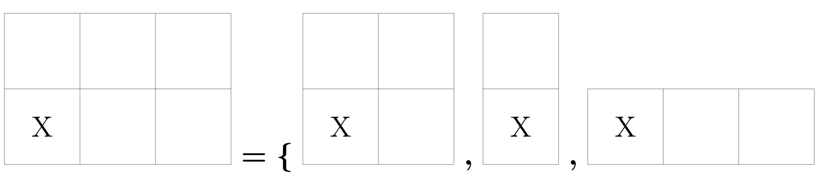
$\mathbf{\} \equiv \{*0, *1, *2\} \equiv *3}$



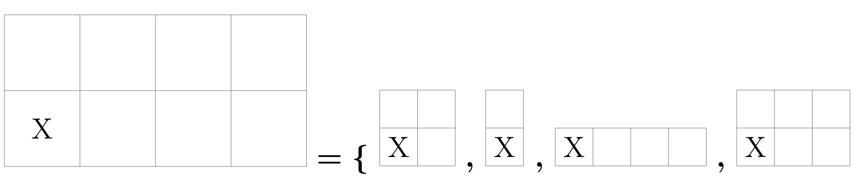
$\mathbf{\} \equiv \{*0\} \equiv *1}$



$\mathbf{\} \equiv \{*1\} \equiv *0}$



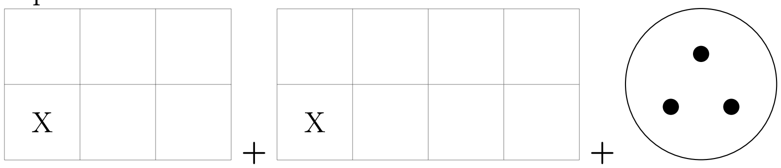
$\mathbf{\} \equiv \{*0, *1, *2\} \equiv *3}$



$\mathbf{\} \equiv \{*0, *1, *3\} \equiv *2}$

Nimber equivalents in sums

Example:

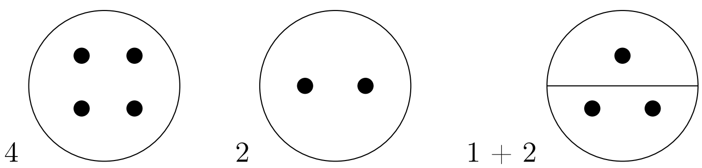


Chomp Chop Nim

This is just like a game of Nim, we just need to balance them.

$\equiv \boldsymbol{*4 + *2 + *3}$
$\equiv \boldsymbol{*(4 \oplus 2 \oplus 3) \equiv *5}$ (like binary addition)
$\equiv \boldsymbol{*(4) + *(\cancel{2}) + *(\cancel{2} + 1) \equiv *5}$ (since we cancel the balanced nimbers)

010_2
011_2
$\underline{100_2}$
$101_2 = 5_{10}$ (five in decimal)



We need to look for the highest odd power of 2, a *1 is needed in the 1^{st} stack to balance with the last stack.

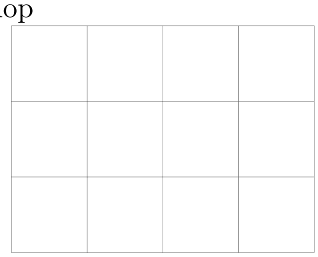
$\boldsymbol{*1 + *2 + *3 \equiv *\cancel{1} + *\cancel{2} + *(\cancel{1} + \cancel{2}) \equiv *0}$ (Every balanced Nim game is of type P)

PUB

Theorem: Let $\boldsymbol{n = 3l + k}$ where $\boldsymbol{0 \leq k \leq 2}$ then a PUB position of n bricks is equivalent to $\boldsymbol{*k}$

$\boldsymbol{P_n = \{P_{n-1}, P_{n-2}\}}$
$\boldsymbol{P_0 \equiv *0}$
$\boldsymbol{P_1 \equiv *1}$
Every position is one of these: $*0, *1, *2$, or (pos mod 3)

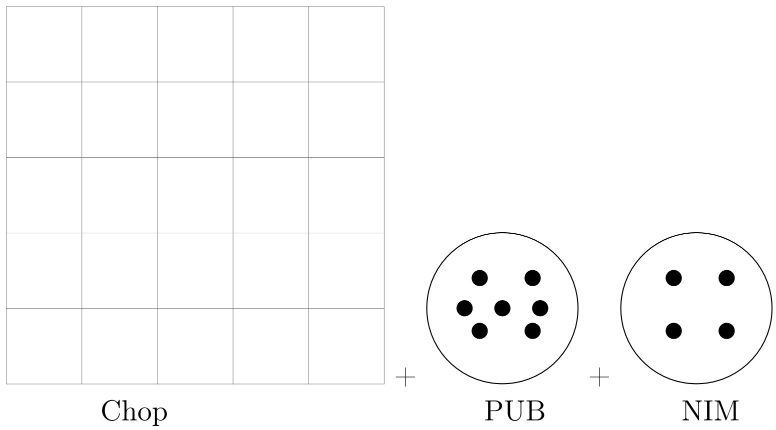
Chop



$m \times n$ grid

Theorem: For every m, n ≥ 1, an $m \times n$ position in chop is equivalent to $*(m-1) + *(n-1)$

Example:



Chop: $\boldsymbol{*3 + *4 \equiv *(2+1) + *(4) \equiv *7}$
PUB: 7 bricks $\Rightarrow = *1$
Nim: $\boldsymbol{*4}$

Sum $= \boldsymbol{*(7+1+4) \equiv *(\cancel{4}+2+\cancel{1}) + *(\cancel{1}) + *(\cancel{4}) \equiv *2}$
(crossing out the balanced nimbers, since $\boldsymbol{*5 + *1 + *4 \equiv *0}$, note $\boldsymbol{*5 \equiv *(4+1)}$)

Partisan games:

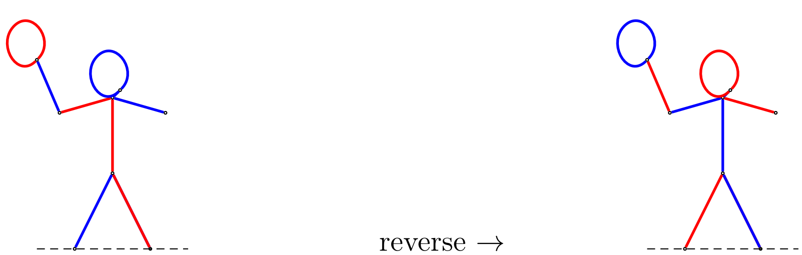
- Sprague - Grandy Therom for impartial games
- Analogus for partisan

Game called: Hackenbush (Hb) (It's Normal play game between R and L, where last player always win if both play optimally). The game board is a graph with edges that are of 2 colors. Some are attached to the ground.

L can cut blue; R can cut red; Last player with a edge wins.
After the edge is erased, any part of the graph not connected to the ground is removed.

Definition: $\boldsymbol{\cdot 0}$ is the Hb position with no edges (no moves left & type P)

The sum operator and equivalence still apply: $\boldsymbol{\alpha + \cdot 0 \equiv \alpha}$ ($\boldsymbol{\cdot 0}$ behaves like $\boldsymbol{*0}$)



Negation: Reverses the colors which inverts the positions and their advantages/disadvantages.

Prop. If α and β are Hb positions

1. $\boldsymbol{-(-\alpha) \equiv \alpha}$
2. $\boldsymbol{\alpha + (-\alpha) \equiv \cdot 0}$
3. $\boldsymbol{\beta + (-\alpha) \equiv \cdot 0 \implies \alpha \equiv \beta}$

Integer Positions

For every positive integer n, define ·n to be the position consisting of n isolated black edges

$\cdot(-2)$ $\cdot(-1)$ $\cdot 0$ $\cdot 1$ $\cdot 2$

$\cdot 3$ negate to $\cdot(-3)$

How would addition work?

Theorem: For any integer m and n:

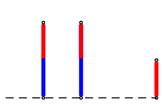
1. $\boldsymbol{-(\cdot n) \equiv \cdot(-n)}$
2. $\boldsymbol{(\cdot m) + (\cdot n) \equiv \cdot(m + n)}$

Who has the advantage?

- $n > 0$, L has advantage
- $n < 0$, R has advantage
- For a given $\cdot(n)$, $(-(\cdot n))$ says the person has an advantage of n.

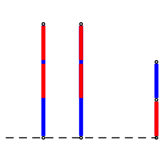
Fractional Positions

- an advantage of $\boldsymbol{\frac{1}{2} \equiv \alpha}$
  $\alpha + \alpha + \cdot(-1) \equiv \cdot 0$
  same as $\boldsymbol{\cdot\frac{1}{2} + \cdot\frac{1}{2} + \cdot(-1) \equiv \cdot 0}$ where $\boldsymbol{\alpha \equiv \cdot\frac{1}{2}}$



$\alpha + \alpha + \cdot(-1)$

- an advantage of $\boldsymbol{\frac{1}{4}(\beta)}$
  $\beta + \beta + \cdot(-\alpha) \equiv \cdot 0$ same as $\boldsymbol{\cdot\frac{1}{4} + \cdot\frac{1}{4} + \cdot(-\frac{1}{2}) \equiv \cdot 0}$ where $\boldsymbol{\beta \equiv \cdot\frac{1}{4}}$ and $\boldsymbol{\alpha \equiv \cdot\frac{1}{2}}$



$\beta + \beta + \cdot(-\alpha)$

Formalize these positions

- $\mathbf{\frac{1}{2^k}}$ is the Hackenbush advantage where there are k lines connected from an opponent base. L has an advantage of $\cdot\mathbf{\frac{1}{2^k}}$
- Lemma: For every position integer k, $\cdot\mathbf{\frac{1}{2^k}} + \cdot\mathbf{\frac{1}{2^k}} \equiv \cdot\mathbf{\frac{1}{2^{k-1}}} \implies \cdot\mathbf{\frac{1}{2^k}} + \cdot\mathbf{\frac{1}{2^k}} + \cdot(-\mathbf{\frac{1}{2^{k-1}}}) \equiv \cdot 0$

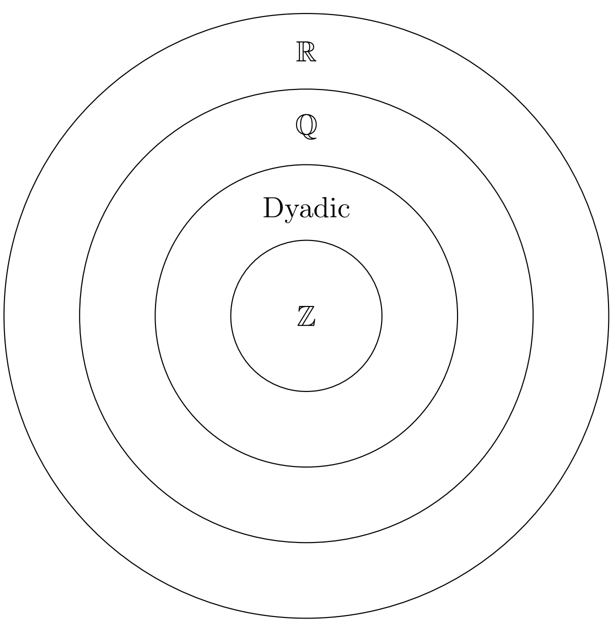
Dyadic Number: Any number that can be expressed as a fraction where the numerator is an integer and the denominator is a power of 2.
Examples:

$$\frac{17}{32} \qquad \frac{531}{16} \qquad \frac{15}{64}$$

- Every dyadic number has a unique (finite) binary expansion.
- For every $\frac{m}{2^k}$ dyadic number, -($\frac{m}{2^k}$) is also a dyadic number.
- The sum of two dyadic numbers is a dyadic number

$$\begin{aligned}\frac{83}{64} &= \frac{1}{64}(83)\\ &= \frac{1}{64}(64+19)\\ &= \frac{1}{64}(64+16+3)\\ &= \frac{1}{64}(64+16+2+1)\\ &\quad \textit{distribute}\\ &= 1 + \frac{1}{4} + \frac{1}{32} + \frac{1}{64}\end{aligned}$$

A Map of the number sets



Birthdays of Numbers

Day 0								0							
Day 1							-1	0	1						
Day 2					-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2				
Day 3	-3	-2	$-\frac{3}{2}$	-1	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{3}{2}$	2	3