Games, Puzzles, & Computation

CSCI 4341 notes for 2/20/17 - 2/25/17

Group F

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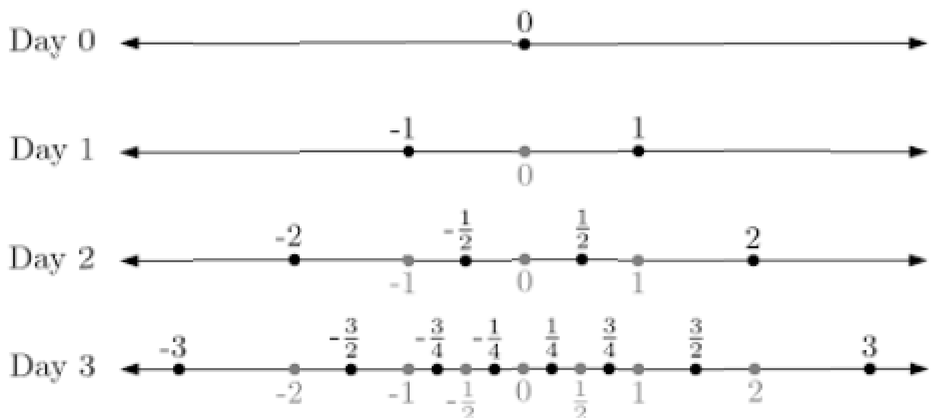
Dyadic numbers have birthdays! Yes!

Numbers were generated from Day 0 onwards, each day creating new numbers. But if we're generating these, what happened at Day 0?

Let's say for each day n, the next figure will show in black the new numbers born on day n, while the numbers born on days before n are gray.



So when is ⅓ created? In infinity days. (It won't get there)

General Procedure: On day 0, the number 0 is born. If $a_1 < a_2 < \ldots\ldots < a_l$ are the numbers born on days, 0, 1,......., n_l then on day n+1 the following new numbers are born:

- The largest integer less than a_1 (if -2 on day 2, -3 now)
- The smallest integer greater than a_1 (if 2 on day 2, 3 now)
- The number$(a_i + a_{i+1})/2$ for every $1 <= i <= l - 1$

Proposition: Every open interval of real numbers (a,b), (a, ∞), (- ∞, b), or even (- ∞,∞), contains a unique oldest dyadic number.

{0, | *1} = ↑

In this Up position,

L can move to type P position

R can move to star 1

In the Down position,

{*1 | 0} = ↓

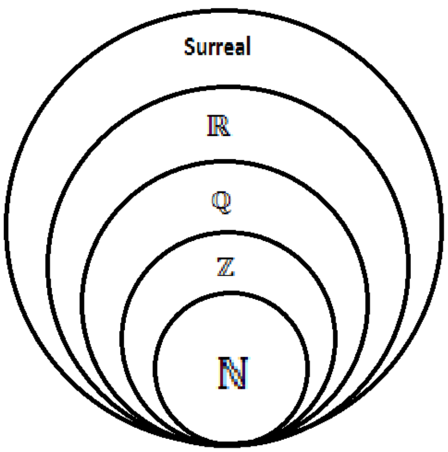
R can move to type P position

L can move to star 1

The real question is why is this the case when ½ = {-1 0 | 1} ?

What would we expect that position to be able to do?

In here, she has the advantage either way. If he moves, she gets an advantage of 1, but if she moves ½ she'll have the advantage after he moves to 1.



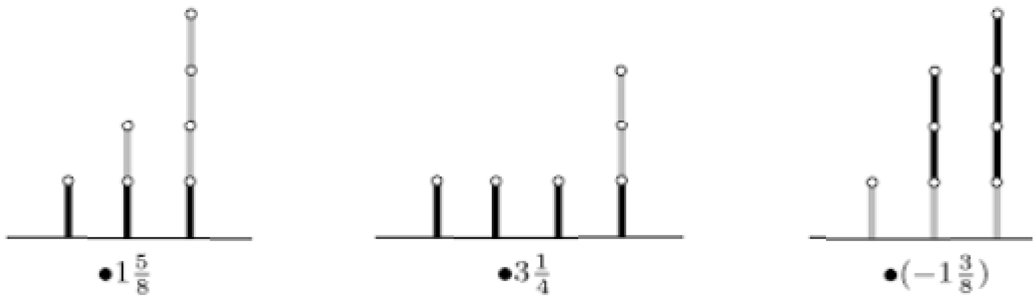
Why are surreal numbers surrounding real numbers and the rest?

For every real number, there is a surreal number, but not all surreals have a real number equivalence

For more information on the number universe, the book recommends the eager learner to look into *Winning Ways for Your Mathematical Plays*. On amazon.com for $68.

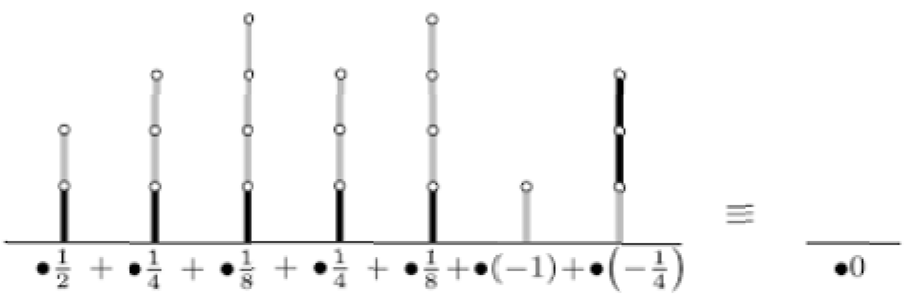
Dyadic positions - Hackenbush positions associated with a dyatic number.

For every dyadic number q > 0 with binary expansion $2^{d_1} + 2^{d_2} + \cdots + 2^{d_\ell}$ we define the position $\bullet q = \bullet 2^{d_1} + \bullet 2^{d_2} + \cdots + \bullet 2^{d_\ell}$.



To get negative positions, we define •(-q) = -(•q).

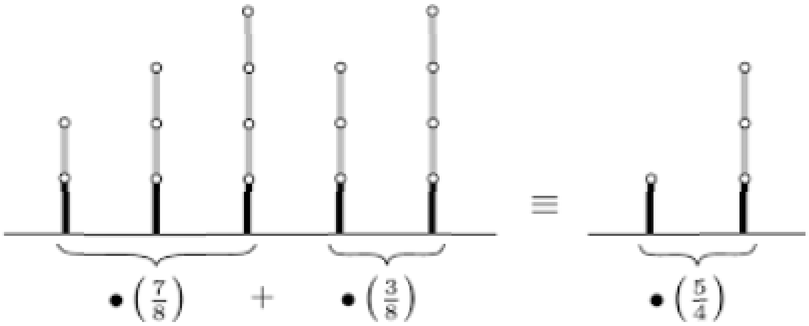
Lemma 4.8. *Let $a_1, \ldots, a_n$ be numbers, each of which is either 0 or has the form $\pm 2^k$ for some integer k. If $a_1 + a_2 + \cdots + a_n = 0$, then $\bullet a_1 + \bullet a_2 + \cdots + \bullet a_n \equiv \bullet 0$.*



Theorem - If p,q are dyadic numbers

1. -(•p) ≡ •(-p)
2. (•p) + (•q) ≡ •(p+q)

Proof – The first part of this theorem follows from the definitions of negation and our dyadic positions. For the second part, apply the previous lemma to obtain $\bullet p + \bullet q + (-\bullet(p+q)) \equiv \bullet 0$. Then adding $\bullet(p+q)$ to both sides of this equation yields the desired result.



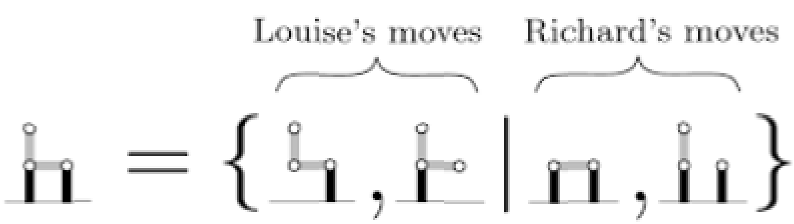
Observation. For a dyadic number q

$$\bullet q \text{ is type} \begin{cases} \text{L} & if \quad q > 0, \\ \text{P} & if \quad q = 0, \\ \text{R} & if \quad q < 0. \end{cases}$$

Adding two positions α ≡ •q and β ≡ •p, which offer an advantage of q and p yield a position which gives L the advantage of p + q.

Before we saw the MEX Principle, which gave us a recursive procedure to determine the nimber equivalent of any position in an impartial game.

Now, we're introducing the Simplicity Principle, which will provide a procedure to determine a dyadic position equivalent to a given one under certain assumptions.



The proof of the Simplicity Principle involves the basic properties of dyadic positions. We prove this by using position notation, for every k > 0 we have

$$\bullet\frac{1}{2^k} = \left\{\bullet 0 \mid \bullet 1, \bullet\frac{1}{2}, \ldots, \bullet\frac{1}{2^{k-1}}\right\}.$$

So, if a player moves $\bullet\frac{1}{2^k}$ to $\bullet c,$ then either Richard moved and c is at least $\frac{1}{2^k}$ larger than $\frac{1}{2^k}$, or Louise moved and c is $\frac{1}{2^k}$ smaller. This principle yields the following Lemma:

Lemma 4.11. *Let* $c = \frac{n}{2^k}$ *with* $k \geq 1$ *and suppose a player moves the position* $\bullet c$ *to the new position* $\bullet c'$.

(1) *If Louise moved, then* $c' \leq c - \frac{1}{2^k}$.

(2) *If Richard moved, then* $c' \geq c + \frac{1}{2^k}$.

The Simplicity Principle.

Consider a position in a partisan game given by $\gamma = \{ \alpha_1, \dots, \alpha_m \mid \beta_1, \dots, \beta_n \}$ then suppose:

$\alpha_i \equiv \bullet a_i$ *for* $1 \leq i \leq m$,

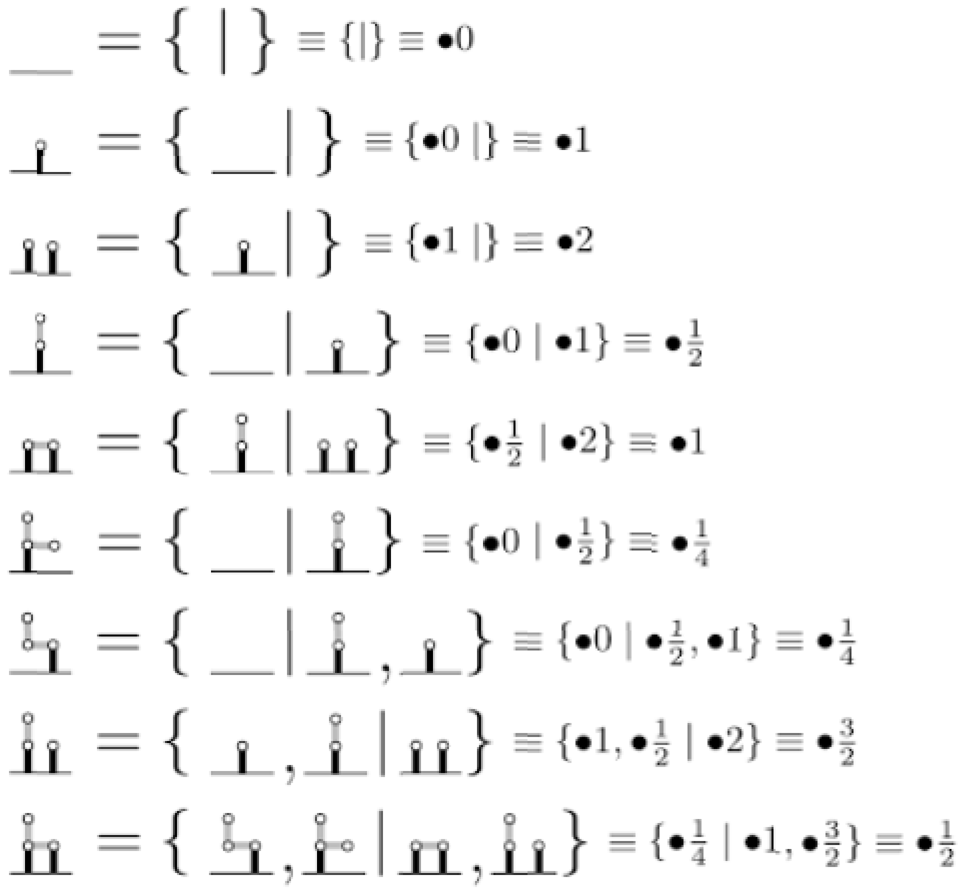
$\beta_j \equiv \bullet b_j$ *for* $1 \leq j \leq n$.

If there do not exist a_i *and* b_j *with* $a_i \geq b_j$, *then* $\gamma \equiv \bullet c$ *where c is the oldest number larger than all of* $a_1, \dots, a_m$ *and smaller than all of* $b_1, \dots, b_n$.

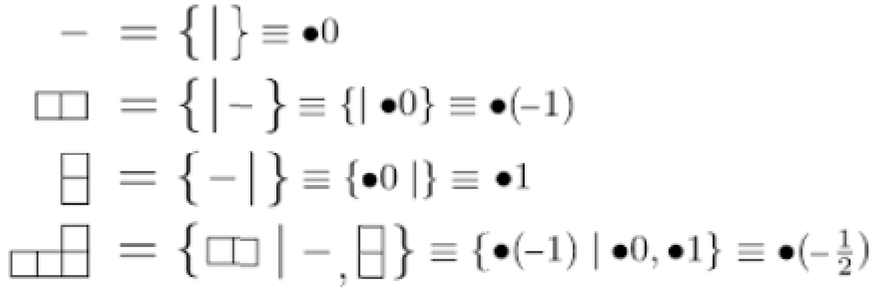
Proof. Assume first that both players have available moves and (by possibly reordering) that $a_m \leq \cdots \leq a_1$ and $b_1 \leq \cdots \leq b_n$. Then c is the oldest number in the interval (a_1, b_1).
To simplify, assume further that $c = \frac{\ell}{2^k}$ where $\ell, k \geq 1$ (the other cases are similar). Since both $c + \frac{1}{2^k}$ and $c - \frac{1}{2^k}$ are older than c, we have

$$c - \frac{1}{2^k} \leq a_1 < c < b_1 \leq c + \frac{1}{2^k}.$$

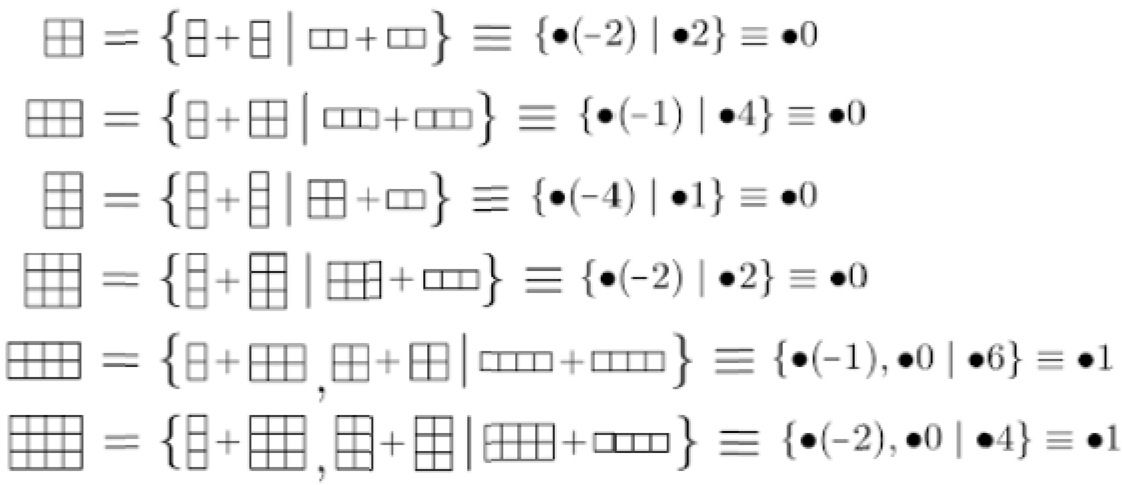
Ex:



Example: Domineering (L = ↕ and R = ↔)



CUT CAKE (L = ↕ and R = ↔)



Sums of Positions:

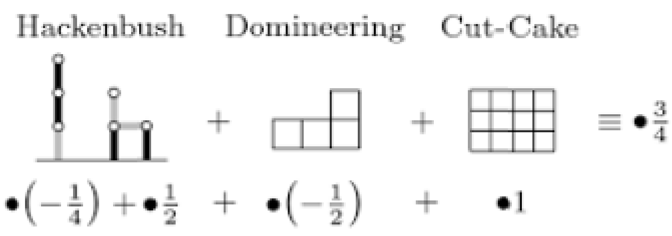


Figure 4.19. Sums of positions

Summary:

- For normal play games
  - Nimbers
  - Dyadic positions
- MEX Principle → any position in an impartial game is equivalent to a nimber.
- Simplicity Principle → some positions in partisan games are equivalent to dyadic positions.

  Allows us to understand sums.
  - Two positions equivalent to nimbers *a and *b sum
    - Sum is equivalent to *(a⊕b) NOTE: Nim – Sum
  - Two positions equivalent to dyadic position
    - Sum is equivalent to ·(a+b)