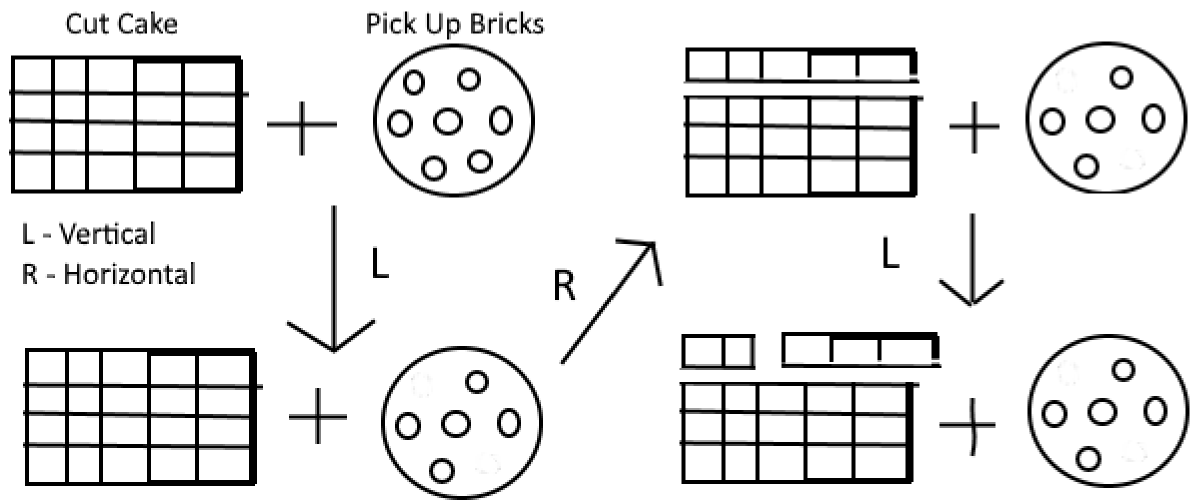
1 2/15/17

1.1 Sums of Position

- already used sums of component

Def'n: δ, α, β are positions in normal play games. Define $\alpha + \beta$ to be a new position consisting of components α and β. To move in $\alpha + \beta$, a player chooses a component to move in.

- A player moves from $\alpha + \beta$ to either $\alpha' + \beta$ or $\alpha + \beta'$.



1.2 Determinate Sums

What is our strategy/ how do types behave under sums? Determine the type of the sum based on the type of the components.

Ex. Let α be some position in some game of type R
α + P∪B

What is R's move?

- Since the P∪B game is type P, it doesn't change anything.
- Ignore P∪B game until L makes a move. Then respond in that component.
- Normal play games - last move wins, so next player plas in other comp.

Prop. If β is type P, then α and $\alpha + \beta$ are the same type.

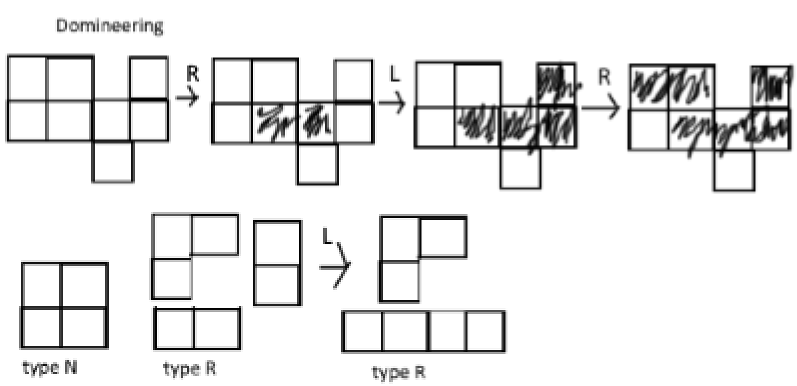
Prop. If α and β are both type L(R) then $\alpha + \beta$ is type L(R).

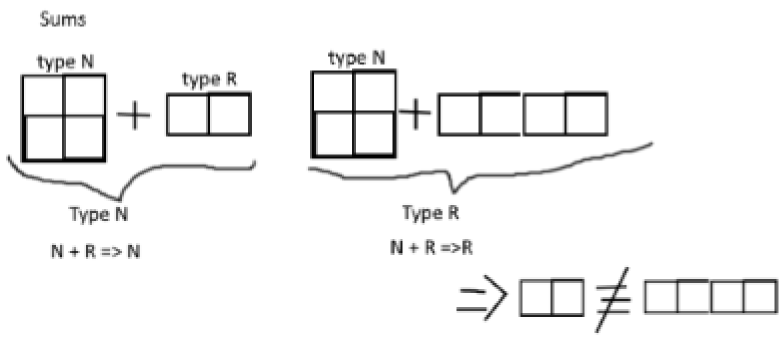
T	L	R	N	P
L	L	?	?	L
R	?	R	?	?
N	?	?	?	N
P	L	R	N	P

1.3 Intermediate Sums

Domineering - cover some grid with dominoes which cover 2 spots at once.

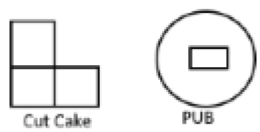
Assume R is horizontal, and L is vertical.





1.4 Equivalence

Move beyond one game at a time



The next player to move will win either game.

Def'n: Two positions α and α' in (possibly different) normal play games are equivalent if for every position β in any normal play games, the two positions $\alpha + \beta$ and $\alpha' + \beta$ have the same type.

Equivalence Relations

Prop. If α, β, δ are positions in normal play games.

1. $\alpha \equiv \alpha$ (reflexivity)
2. $\alpha \equiv \beta \rightarrow \beta = \alpha$ (symmetry)
3. $\alpha \equiv \beta$ and $\beta \equiv \delta \rightarrow \alpha \equiv \delta$ (transitivity)

How does equivalence relate to type?

- If two positions are equivalent $\rightarrow$ they have the same type

Prop. If $\alpha \equiv \alpha'$, then they have the same type.

- Let β be a position in a normal play game with no moves left.
  $\alpha \leftrightarrow \alpha + \beta \leftrightarrow \alpha' + \beta \leftrightarrow \alpha'$

- Algebra w/ +, $\equiv$
- Prop. If α, β, δ are positions in normal play games.
  1. $\alpha + \beta \equiv \beta + \alpha$ (commutably)
  2. $(\alpha + \beta) + \delta \equiv \alpha + (\beta + \delta)$ (associatably)

Lemma. Given position α, β,δ in normal play games

1. If $\alpha \equiv \alpha'$, then $\alpha + \beta \equiv \alpha' + \beta$
2. If $\alpha_i \equiv \alpha_i$' for 1 $\leq$ i $\leq$ n, then $\alpha_i + ... + \alpha_n \equiv \alpha_i$' + α_n'
3. If $\alpha_i \equiv \alpha_i$ for 1 $\leq$ i $\leq$ m and $\beta_j \equiv \beta_j$' for 1 $\leq$ j $\leq$ n, then $\{\alpha_n$,α_m — β_1,...,β_n $\} \equiv \{\alpha_1$,...,α_n' — β_1,...,β_n $\}$

Type P is equivalent to zero under normal addition.
Lemma. If β is type P, then $\alpha + \beta = \alpha$

Prop. If α and α' are type P, then $\alpha \equiv \alpha$' $\rightarrow \alpha + \delta \equiv \delta \equiv \alpha$' + δ

Lemma. If $\alpha + \beta$ and α' + β are both type P, then $\alpha \equiv \alpha$'

1.5

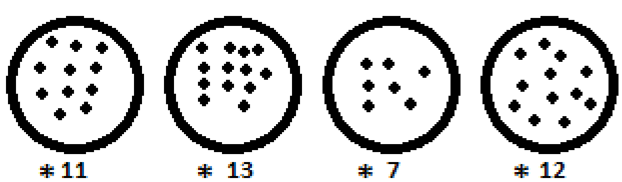
Cor. Every position in an impartial game is one of the following:

- Type N - The next player to play has a winning strategy.
- Type P - The previous (or second) player to play has a winning strategy.

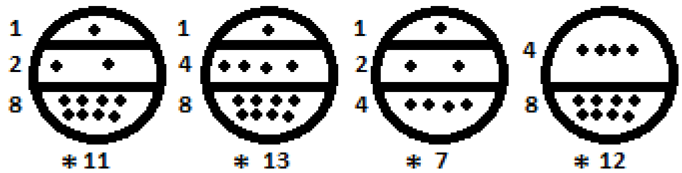
Cor. A position in an impartial game is:

- Type N - If $\exists$ a move to a position of type P.
- Type P - If there is no move to a position of type P.

For convenience, # of stones in each pile



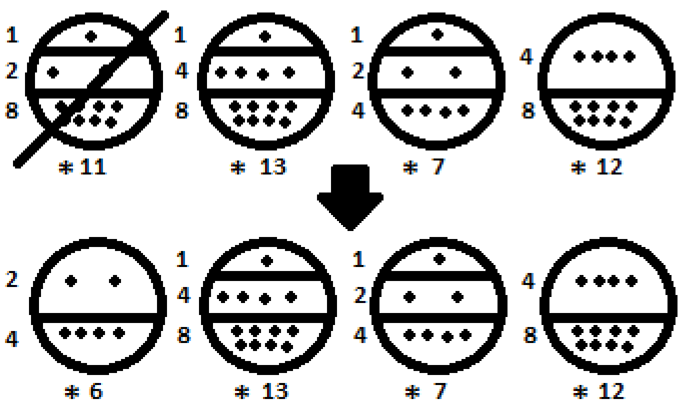
Def'n: Given a Nim position $*\alpha_1 + *\alpha_2 + ... + *\alpha_k$, it is balanced if, for every power of 2, the total # of subpiles of that size is even.



$2^0 + 2^1 + 2^2 + ... + 2^n < 2^{n+1}$

Procedure - Balance an unbalanced position:

- Let $*\alpha_1 + *\alpha_2 + ... + *\alpha_k$ be an unbalanced game of Nim.
- Suppose 2^n is the largest power of 2 which there are an odd # of subpiles Curretly all subpiles of $2^i, j > m$are balanced for every j ¡ m, if there are an odd # of subpiles of S^j, excluding this pile, leave s^i stones. Since at least 2^m stones could be removed, and $2^0 + 2^i + ... + 2^{n-1} < 2^m$ stones should be left, this is a legal move.



Prop. Every balanced Nim position is type P and every unbalanced Nim position is type N.

1.6 Nimbers

Its helpful to think of Nim positions like #'s

- we call a stack $*\alpha$ a nimber
- Based on prop and that P doesn't additively change anything a type P position *0
- $\alpha \equiv \gamma$
- every balanced position $\gamma \equiv *0$

Imagine the 2^{nd} player could add a stack to the game before it starts
Original $*a_1 + ... + *a_L$

- add *b
- to a balanced position of type P to determine odd # of subpiles of 2^j

$*a_1 + ... + *a_l + *b = *0$