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 Group F

Numbers have birthdays! Yes!

Numbers were generated from Day 0 onwards, each day creating new numbers.

But if we're generating these, what happened at Day 0?

At Day 0: 0 entered the universe

At Day 1: -1 0 1 was the new range, adding to the 0 that was there from the day before

At Day 2 the new range: -2 -1 -1/2 0 1/2 1 2

At Day 3: -3 -2 -(3/2) -1 -1/2 -1/4 0 (3/4) 1 (3/4) (3/2)

Every day, a new number is added from between the number lines

So when is 1/3 created?

In infinity days.

General Procedure: On day 0, the number 0 is born.

If  $a_1 < a_2 < \dots < a_i$  are the #'s born on days,

0, 1, ...,  $n_i$  then on day  $n+1$  the following new numbers are born

- The largest integer less than  $a_1$  (if -2 on day 2, -3 now)
- The smallest integer greater than  $a_1$  (if 2 on day 2, 3 now)
- The  $\#(a_i + a_{i+1})/2$  for every  $1 \leq i \leq n-1$

Proposition: Every open interval of real numbers  $(a,b)$  contains a unique oldest dyadic number.

$(a, \infty)$ ,  $(-\infty, b)$ ,  $(-\infty, \infty)$ .

$\{0, \mid *1\} = \uparrow$

In this Up position,

L can move to type P position

R can move to star 1

In the Down position,

$\{*1 \mid 0\} = \downarrow$

The real question is why is this the case when  $1/2 = \{-1 \ 0 \mid 1\}$

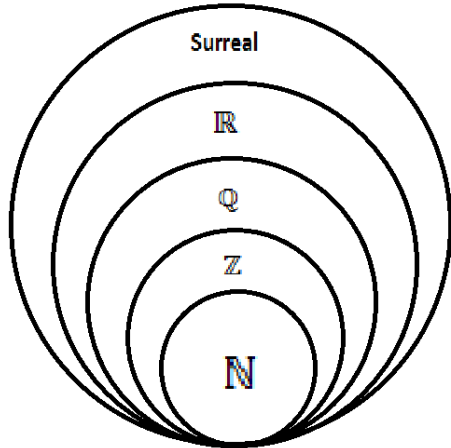
What would we expect that position to be able to do?

In here, she has the advantage either way. If he moves, she gets an advantage of 1, but if she moves 1/2 she'll have the advantage after he moves to 1.

Why are surreal numbers surrounding real numbers and the rest?

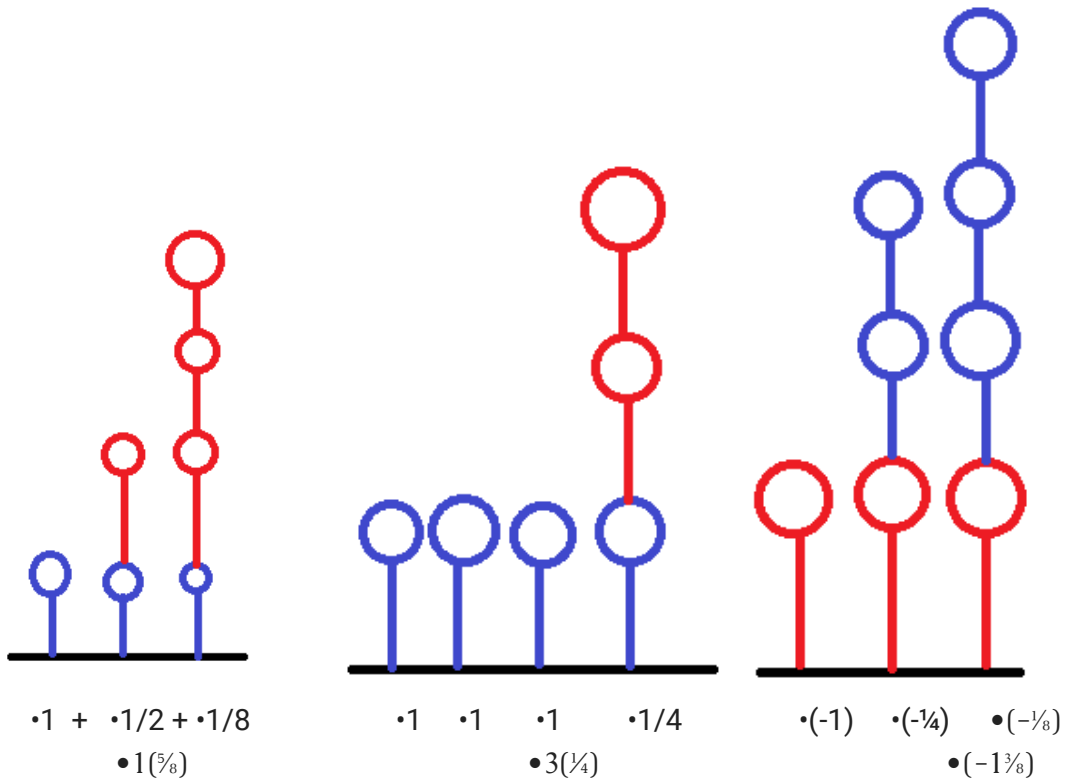
For every real number, there is a surreal number, but not all surreals have a real number equivalence

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Dyadic positions - Hackenbush positions associated with a dyadic number.

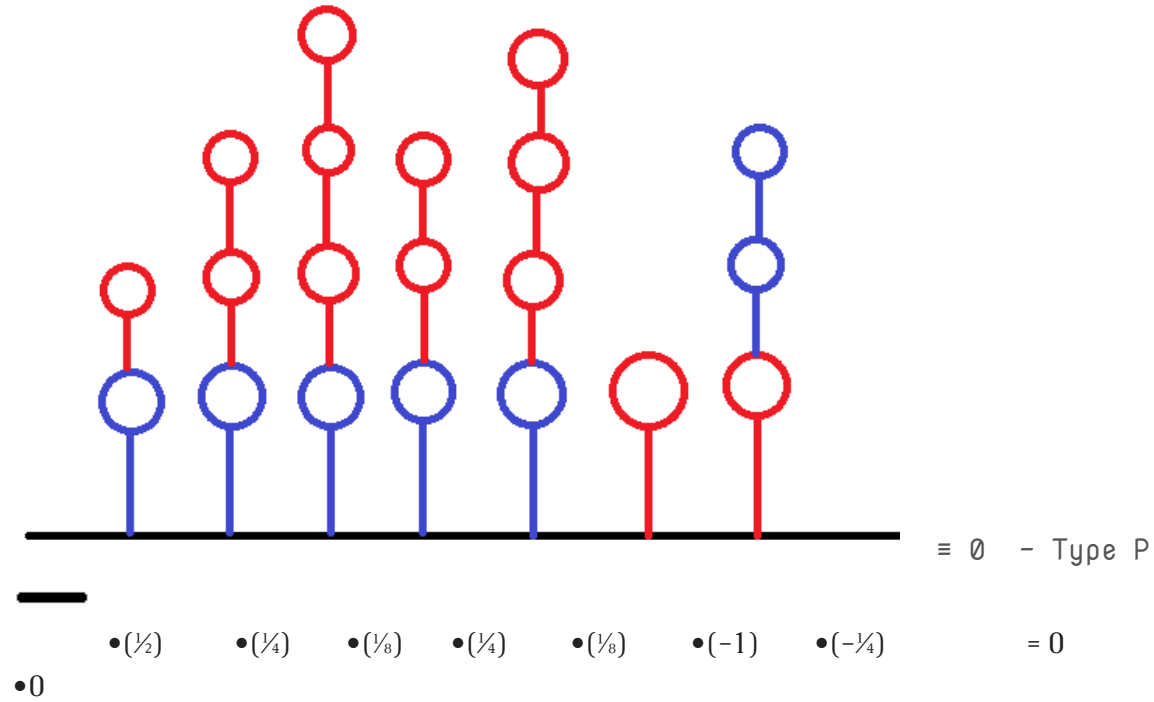
For every dyadic number, there is a position in Hb with a binary  $q > 0$  with binary expansion.  $2^{d_1} + 2^{d_2} + \dots + 2^{d_n}$



$\bullet(-q) = -(\bullet q)$

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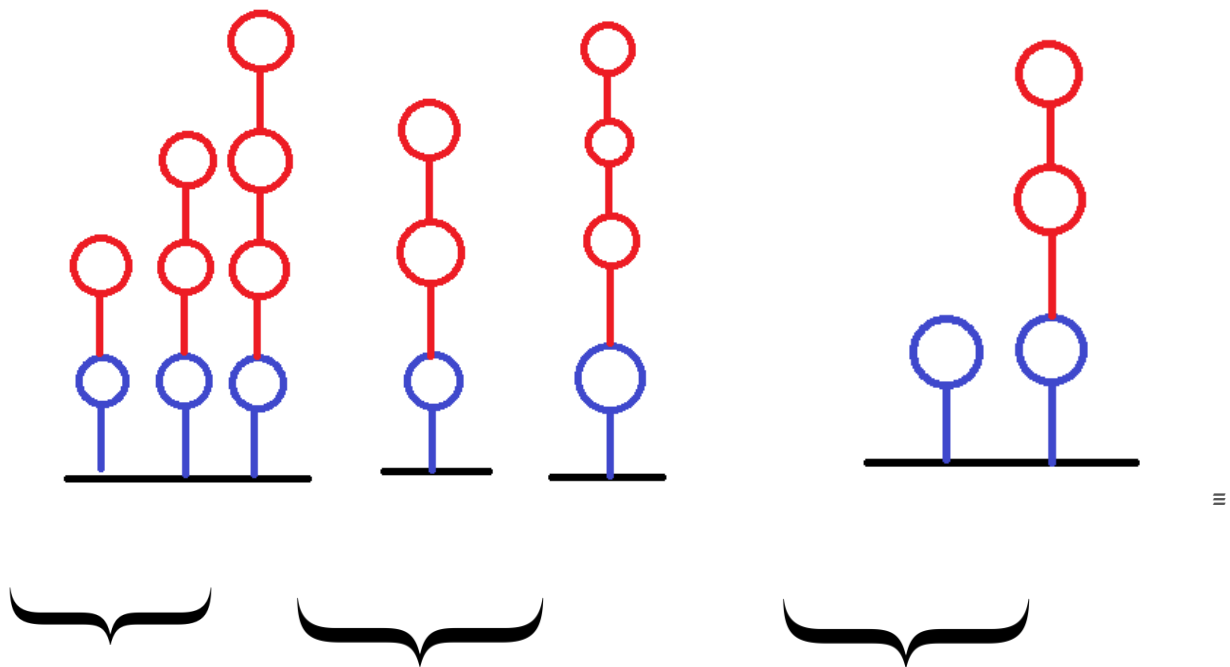
Lemma. Let  $a_1, \dots, a_n$  be numbers, each either 0 or of the form  $\pm 2^k$  for  $k \in \mathbb{Z}$ . If  $a_1 + a_2 + \dots + a_n = 0$ , then  $\bullet a_1 + \bullet a_2 + \dots + \bullet a_n \equiv \bullet 0$



Theorem - If  $p, q$  are dyadic #'s

1.  $\bullet(-p) \equiv \bullet(-p)$
2.  $\bullet(p) + \bullet(q) \equiv \bullet(p+q)$

Example:



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•( $\frac{7}{8}$ )

•( $\frac{3}{8}$ )

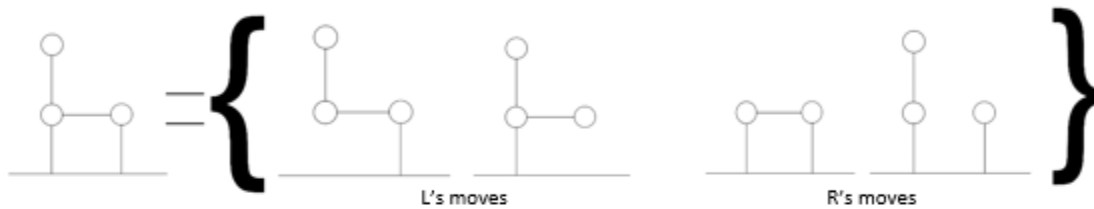
•( $\frac{5}{4}$ )

Obs. For a dyadic #  $q$

• $q$  is  $\left\{ \begin{array}{l} \text{L if } q > 0 \text{ } \leftarrow \text{ She should always win} \\ \text{type} \hspace{15em} \text{P if } q = 0 \\ \text{R if } q < 0 \text{ } \leftarrow \text{ He should always win} \end{array} \right.$

Adding positions  $\alpha \equiv \bullet q$  and  $\beta \equiv \bullet p$  gives L the advantage of  $\bullet(q+p)$  for  $\alpha + \beta$

- Analogous principle for partisan games to the MEX principle for impartial games.
- Procedure to determine dyadic position equivalent.



Basic property of dyadic position.

$$\cdot \frac{1}{2^x} = \{ \cdot 0 \mid \cdot 1, \cdot \frac{1}{2}, \dots, \cdot \frac{1}{2^{x-1}} \}$$

Lemma. Let  $c = \frac{n}{2}k$  with  $k \geq 1$  and suppose a player moves the  $\cdot C$  position  $\cdot C'$ .

1. If L moves, then  $C' < C - \frac{1}{2}k$
2. If R moves, then  $C' \geq C + \frac{1}{2}k$

**The Simplicity Principle.**

Consider a position in a partisan game given by  $\gamma = \{ \alpha_1, \dots, \alpha_m \mid \beta_1, \dots, \beta_n \}$  then suppose:

$$\alpha_i \equiv \cdot a_i \text{ for } 1 \leq i \leq m$$

$$\beta_j \equiv \cdot b_j \text{ for } 1 \leq j \leq n$$

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If there do not exist  $a_i$  and  $b_j$  with  $a_i \geq b_j$  then  $\gamma \equiv \cdot \mathbf{C}$  where  $\mathbf{C}$  is the oldest number larger than all of  $a_i, \dots, a_m$  and smaller than all  $b_j, \dots, b_n$ .  
 $\max(a_i) < C < \min(b_j)$ .

Ex.

$$\underline{\quad} = \{ | \} \equiv \{ 1 \} \equiv .0$$

$$\begin{array}{c} \circ \\ | \\ \underline{\quad} \end{array} = \{ \underline{\quad} | \} \equiv \{ .0 | \} \equiv .1$$

$$\begin{array}{c} \circ \quad \circ \\ | \quad | \\ \underline{\quad} \end{array} = \{ \begin{array}{c} \circ \\ | \\ \underline{\quad} \end{array} | \} \equiv \{ .1 | \} \equiv .2$$

$$\begin{array}{c} \circ \\ | \\ \begin{array}{c} \circ \\ | \\ \underline{\quad} \end{array} \end{array} = \{ \underline{\quad} | \begin{array}{c} \circ \\ | \\ \underline{\quad} \end{array} \} \equiv \{ .0 | .1 \} \equiv \cdot \frac{1}{2}$$

$$\begin{array}{c} \circ \quad \circ \\ | \quad | \\ \begin{array}{c} \circ \\ | \\ \underline{\quad} \end{array} \end{array} = \{ \begin{array}{c} \circ \\ | \\ \begin{array}{c} \circ \\ | \\ \underline{\quad} \end{array} \end{array} | \begin{array}{c} \circ \quad \circ \\ | \quad | \\ \underline{\quad} \end{array} \} \equiv \{ \cdot \frac{1}{2} | .2 \} \equiv \cdot 1$$

$$\begin{array}{c} \circ \\ | \\ \begin{array}{c} \circ \quad \circ \\ | \quad | \\ \underline{\quad} \end{array} \end{array} = \{ \underline{\quad} | \begin{array}{c} \circ \\ | \\ \underline{\quad} \end{array} \} \equiv \{ .0 | \cdot \frac{1}{2} \} \equiv \cdot \frac{1}{4}$$

Example: Domineering (L =  $\uparrow$  and R =  $\leftrightarrow$ )

$$- \equiv \{ | \} \equiv \cdot 0$$

$$\square \equiv \{ | - \} \equiv \{ | \cdot 0 \} \equiv \cdot -1$$

$$\begin{array}{|c|} \hline \square \\ \hline \end{array} \equiv \{ - | \} \equiv \{ \cdot 0 | \} \equiv \cdot 1$$

$$\begin{array}{|c|c|c|} \hline \square & \square & \begin{array}{|c|} \hline \square \\ \hline \end{array} \\ \hline \end{array} \equiv \left\{ \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \mid \begin{array}{|c|} \hline \square \\ \hline \end{array}, - \right\} \equiv \{ \cdot (-1) \mid \cdot 1, \cdot 0 \} \equiv \cdot (-1/2)$$

CUT CAKE (L =  $\updownarrow$  and R =  $\leftrightarrow$ )

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \equiv \left\{ \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \mid \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \right\} \equiv \{ \cdot (-2) \mid \cdot 2 \} \equiv \cdot 0$$

$$\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \equiv \left\{ \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \mid \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array} \right\} \equiv \{ \cdot (-1) \mid \cdot 4 \} \equiv \cdot 0$$

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \equiv \left\{ \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \mid \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \right\} \equiv \{ \cdot (-4) \mid \cdot 1 \} \equiv \cdot 0$$

$$\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} = \left\{ \begin{array}{|c|} \hline \\ \hline \\ \hline \end{array} + \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array} \mid \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \right\} \equiv \{ \cdot(-2) \mid \cdot 2 \} \equiv \cdot 0$$

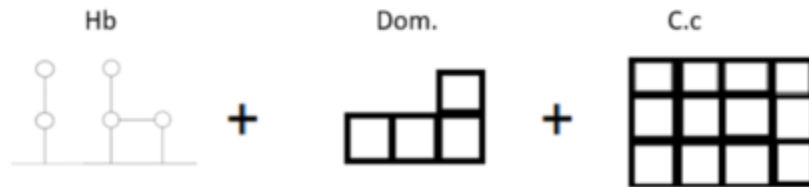
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$$\equiv \{ \cdot 0, \cdot(-1) \mid \cdot 6 \} \equiv \cdot 1$$

$$\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} = \left\{ \begin{array}{|c|} \hline \\ \hline \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array} \mid \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \right\}$$

$$\equiv \{ \cdot(-2), \cdot 0 \mid \cdot 4 \} \equiv \cdot 1$$

Sums of Positions:



$$\cdot(-1/4) + \cdot(-1/2) + \cdot 1 \equiv \cdot 3/4$$



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Summary:

- For normal play games
  - Nimbers
  - Dyadic positions
- MEX Principle → any position in an impartial game is equivalent to a nimber.
- Simplicity Principle → some positions in partisan games are equivalent to dyadic positions.

Allows us to understand sums.

- Two positions equivalent to nimbers  $*a$  and  $*b$  sum
  - Sum is equivalent to  $*(a \oplus b)$  NOTE: Nim – Sum
- Two positions equivalent to dyadic position
  - Sum is equivalent to  $\cdot(a+b)$