A STUDY OF $2 \times n$ AND $3 \times n$ DOMINEERING

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The Game

Domineering is a game played by two players, whom we will call Left and Right, on an $m \times n$ grid. Left and Right alternate turns placing 2×1 dominoes which cover two squares of the grid. Left places these dominoes vertically, and Right places them horizontally.
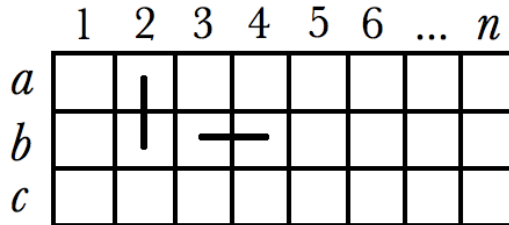

FIGURE 1. A Game of $3 \times n$ Domineering.

In Figure 1, we see a generalized game of $3 \times n$ Domineering. The columns are represented by numbers 1 through n, and the rows are represented by letters a, b, and c. The way we will notate individual moves in this paper is by naming the squares covered by the move. For example, in the game above, Left has moved to a_2b_2 and Right has moved to b_3b_4.

There are four possible outcome classes of Domineering, and these are L, R, N, and P. L denotes a Left player win, R a Right player win, N a first player win, and P a second player win. Note that any game of Domineering will fall in exactly one of these outcome classes. We have determined every outcome class of $3 \times n$ Domineering, and every outcome class of $2 \times n$ except 2×31.

$3 \times n$ Domineering

For $3 \times n$ Domineering, the outcome classes are: $n = 1 \in L, 2 \in N, 3 \in N, n \geq 4 \in R$.

Right's winning strategy is to split any $3 \times n$ game of domineering into a game sum of subgames in R by drawing imaginary vertical lines in between the squares. Since Left

plays vertically, s/he must play in exactly one of these subgames, and Right has a winning response to Left's move in that subgame. Right can continue this strategy until victory.

For 3×1, the outcome class is clearly L because Left has 2 possible moves and Right has 0 possible moves.

For 3×2 the outcome class is N. Any first move Left makes will be a winning move, and $b_1 b_2$ will be a winning first move for Right.

For 3×3, the outcome class is N because any first move that covers b_2 will win the game.

For 3×4, the outcome class is R because if Left goes first, Right can then make a move along the second row like $b_3 b_4$. Now Left has two moves that are mutually exclusive, and Right has two unbreakable moves. Right will win in this case. If Right moves first, he can make a move like $b_3 b_4$, which guarantees two future moves, and Left has at most two more moves. Right will win in this case as well.

For 3×5, the outcome class is R If Left makes the first move, regardless of which move, right can then make a move along the b row which protects two future moves. So after Lefts first move, Right has at least 3 more moves and Left has only 2 remaining. If Right goes first, he will make a move along the b row, and then Left will have at most 3 moves remaining and Right will have at least 3 moves remaining, with Left moving first. Hence, Right will win.

For 3×6, the outcome class is R. If Left moves first, Right can imagine the grid as two 3×3 games, and make the equivalent move in the opposite 3×3 game (by imagining the grid rotated $\frac{\pi}{2}$ and making the same move). If Right moves first he can move to $b_3 b_4$ and in this position Left has only 2 more moves and Right has at least 5 more moves if his second move is also in the b row (he can do that no matter where Left moves), guaranteeing a Right win.

For 3×7, the outcome class is R. If Right goes first, he can win by dividing the grid into a 3×3 subgame and 3×4 subgame, and making his first move in the 3×3 game, which is a first-player win. Left will now lose no matter which subgame s/he plays in because Right has a winning response in the two subgames. If Left goes first, Right will also win. Going through every case to prove why this is true would take a lot of space, but it is clear that Right can win this game by making his first move along the b row, which will preserve 2 future moves.

For $n \geq 7$, n can be expressed as one of the following: $4k + 7, 4k + 6, 4k + 5, 4k$. Right can now divide the game into a game sum of subgames in R (since $n = 7, 6, 5, 4$ are all in R). Since Right will have a winning first move (if Right moves first) and a winning response (if Right moves second) in every subgame, Right will emerge victorious.

$2 \times n$ Domineering

For $2 \times n$ Domineering, the outcome classes are: $n = 1 \in L, 2 \in N, 3 \in N, 4 \in R, 5 \in L, 6 \in N, 7 \in N, 8 \in R, 9 \in L, 10 \in N, 11 \in N, 12 \in R, 13 \in P, 14 \in N, 15 \in N, 16 \in R, 17 \in R, 18 \in N, 19 \in N, 20 \in R, 21 \in R, 22 \in R, 23 \in N, 24 \in R, 25 \in R, 26 \in R, 27 \in N$ For all $n > 27$ (with the possible exception of 2×31), the outcome class of $2 \times n$ Domineering is R.

It is easy to check the outcome class when n is between 1 and 8, so we will not explain these cases, with the exception of 2×4, because this case will be important in the analysis to follow. 2×4 Domineering is a Right win. If Right moves first, a winning first move is a_2a_3. Left has to respond with a_1b_1 or a_4b_4, and Right will block Left's only remaining move with his last move. If Left goes first, two first moves are a_2b_2 and a_3b_3, but any response Right makes will win the game. The other first moves are on either end of the 2×4 grid, leaving a 2×3 grid remaining. Right has a winning response because 2×3 is a first-player win. We can now use the outcome class of 2×4 to prove some useful facts.

Fact 1 $2 \times 4k$ Domineering is in R.

Proof Since 2×4 is a Right player win, we can now say that any $2 \times 4k$ game is a Right win. This is because whenever Left plays, Left must move in one of k 2×4 games which are all in R, meaning Right will have a winning response to Left's move. Right can continue this strategy until s/he wins the $2 \times 4k$ game.

Fact 2 $2 \times (4k + 2)$ Domineering is in $N \cup R$.

Proof This means that Right has a winning strategy playing first. If Right moves first, he should first visualize the game as a sum of one 2×2 subgame and a $2 \times 4k$ subgame. His first move should be in the 2×2 subgame. Now Left must move in the remaining $2 \times 4k$ subgame, and by Fact 1, Left loses.

Fact 3 $2 \times (4k + 3)$ Domineering is in $N \cup R$.

Proof Similarly to the strategy for $2 \times (4k + 2)$, Right should visualize a game sum of 2×3 and $2 \times 4k$ subgames and make the first move in the 2×3 subgame.

Now we will briefly discuss the cases of 2×9, 2×10, 2×11, and 2×12 Domineering. For 2×9, Left's winning first move is a_2b_2. If Right moves first, Left can move either a_2b_2 or a_8b_8, and this is a winning response. For 2×10, we know Right wins moving first by Fact 2. If Left goes first, the winning move is a_2b_2. Since both players win moving first, 2×10 is in N. Right also wins 2×11 moving first by Fact 3, and Left's winning first move is a_2b_2, so the game is in N. 2×12 is in R by Fact 1.

Now we will examine 2×13, which will be important to our analysis because it is the only $2 \times n$ game in P, and because the numbers 4 and 13 are relatively prime. This fact will allow Right to win $2 \times n$ Domineering when n can be expressed as a linear combination of 4 and 13 and the coefficient of 4 is at least 1, as we will explain later on. If Left moves first, s/he cannot win, and we will not go through every position of the game to show why this is true. The reader can check it without too much difficulty by examining all possible Left moves and the ensuing positions. If Right moves first, Left will be able to move in $a_2 b_2$ or $a_{12} b_{12}$. Let the resulting game be G.

FIGURE 2. The game G.

We can prove $G \geq (2 \times 9)$. Without loss of generality, assume that Right moves first to $a_{10} a_{11}$ and Left moves to $b_1 b_2$. If Right never makes the moves $b_9 b_{10}$ or $b_{11} b_{12}$, Left can win $G - (2 \times 9)$ by symmetry going second.

FIGURE 3. The Game $G - (2 \times 9)$.

Now assume Right does make the move $b_9 b_{10}$. In this case, Left should move to $b_7 b_8$ in $-(2 \times 9)$. Then, when Right moves to $b_{11} b_{12}$ in G, Left can use the protected move in G s/he created by moving to $a_2 b_2$, which is $a_1 b_1$. Left is still in a position to win by symmetry. This shows that $G - (2 \times 9) \geq 0$ and therefore $G \geq (2 \times 9)$. Since 2×9 is a Left win, G is also a Left win. Furthermore, no matter where Right moves first, Left can create a version of G on his/her first move. Therefore $2 \times 13 \in P$.

Now we will try to combine 2×4 games and 2×13 games to create bigger games. If n can be expressed as $4x + 13y$ with $x > 0, y \geq 0$, then it will be a Right player win since Right player can visualize the game as a game sum of games in R and P, the sum of which will be less than 0 (in other words, a Right win). Let's determine which n can be expressed in this way.

$$n = 4x + 13y$$

Let's look at this equation modulo 4. We get this: $n \equiv y \pmod 4$. If $n = 4k + 3$, it follows that $n \equiv 3 \pmod 4$. By substitution, $y \equiv 3 \pmod 4$. Since our constraints dictate $y \geq 0$, it follows that $y \geq 3$, and $n \geq (4 \cdot 0 + 13 \cdot 3) = 39$.

It is logical to now consider $n = 36, 37, 38, 39$ and show they are all in R.

$2 \times 36 \in R$. This follows immediately from Fact 1.

$2 \times 37 \in R$. $37 = 6 \cdot 4 + 13$, so Right has a winning strategy.

$2 \times 38 \in R$. $38 = 4 \cdot 3 + 2 \cdot 13$, so Right has a winning strategy.

$2 \times 39 \in R$. Since $39 = 3 \cdot 13$, Right can win can going second, and since 39 can be expressed as $4k + 3$, we know that Right can win going first. Hence the game is a Right win.

When $n > 39$, n can be expressed by adding $4k$ to either 36, 37, 38, or 39, which are all in R. By this reasoning, $2 \times n$ Domineering is a Right player win when $n \geq 39$.

This line of reasoning is also useful for several cases when $14 \leq n \leq 35$. These are the cases (note: the cases when $n = 16, 20, 24, 28, 32$ were shown to be in R by Fact 1, but we include them here because they can also be expressed as $4x + 13y$ with $x > 0, y \geq 0$).

2×16: $16 = 4 \cdot 4 + 13 \cdot 0$

2×17: $17 = 4 \cdot 1 + 13 \cdot 1$

2×20: $20 = 4 \cdot 5 + 13 \cdot 0$

2×21: $21 = 4 \cdot 2 + 13 \cdot 1$

2×24: $24 = 4 \cdot 6 + 13 \cdot 0$

2×25: $25 = 4 \cdot 3 + 13 \cdot 1$

2×28: $28 = 4 \cdot 7 + 13 \cdot 0$

2×29: $29 = 4 \cdot 4 + 13 \cdot 1$

2×30: $30 = 4 \cdot 1 + 13 \cdot 2$

2×32: $32 = 4 \cdot 8 + 13 \cdot 0$

2×33: $33 = 4 \cdot 5 + 13 \cdot 1$

2×34: $34 = 4 \cdot 2 + 13 \cdot 2$

Because $n = 4x + 13y$ in these cases, these games are all in R.

Now there are only 10 out of n cases that remain to be discussed. They are $n = 14, 15, 18, 19, 22, 23, 26, 27, 31, 35$.

Let's first examine 2×14. We know the outcome class is N or R, by Fact 2. If Left moves first, the winning move is $a_1 b_1$, which leaves a 2×13 game remaining, in which Right must move. Since $2 \times 13 \in P$, Right will lose, meaning Left wins moving first. Therefore $2 \times 14 \in N$.

2×15 is also either in N or R, by Fact 3. Left has a winning move going first: a_2b_2. Now the game is a game sum of 2×1 and 2×13. Right has to move in 2×13, and he will lose. So $2 \times 15 \in N$.

2×18 is in N or R by Fact 2. Left has a winning first move in a_2b_2. No matter where Right moves, Left can move either a_3b_3 or $a_{18}b_{18}$, creating a situation like we saw in 2×13 after 2 moves. By the same method, we can show that the game is greater than or equal to $(2 \times 13) = 0$. It follows that Left will win moving first, so $2 \times 18 \in N$.

2×19 is in N or R by Fact 3. Left can win by moving to a_9b_9, creating a game sum of 2 2×9 games. Since $2 \times 9 \in L$, Left can win 2×19 going first, hence $2 \times 19 \in N$.

2×23 is in N or R by Fact 3. Left can win by moving $a_{10}b_{10}$, creating a sum of 2×9 and 2×13 which are in L and P, respectively. Wherever Right moves, Left has a winning response. Since Left can win moving first, we have $2 \times 23 \in N$.

2×26 is in R because if Left goes first, Right can view the game as a sum of two 2×13 games, and if Right goes first the winning move is $a_{13}a_{14}$. This move effectively allows Left to only play in two games of 2×12, which is in R.

2×27 is in N or R by Fact 3. Left can win by moving $a_{14}b_{14}$, creating 2 2×13 games, both second-player wins. Thus, $2 \times 27 \in N$.

The cases remaining are 2×22, 2×31, and 2×35. We have not proved that these cases are Right player wins, but using the Combinatorial Game Suite software we have computed that 2×22 is in R. It follows that 2×35 is in R because Right can view the games as a sum of 2×22 and 2×13, which are in R and P, respectively.

2×31 was more tricky. It is large enough that our software could not compute its outcome class, and our attempts to analyze the game in other ways were fruitless. By Fact 3 we know it is in either R or N, but we cannot show anything else. However, the CGSuite software did compute that the only sensible Left moves in 2×31 are a_2b_2 and a_4b_4, so further analysis of the positions resulting from those moves might be worthwhile. We conjecture that it is a Right win.

This concludes our study of $2 \times n$ and $3 \times n$ Domineering.