What is a game?

- ?
- Our definition captures many games people play
  - board games - chess, go, etc
  - card games - Poker, bridge
  - one-player puzzles - Rush hour, peg solitaire, sliding blocks
  - zero-player games - automata / simulations - Conway's game of life

- Four main features
  1. positions - finite board configurations, card distributions (bounded state) - finitely amt. information
  2. players - players take turns. has clear goals and clear list of allowable moves, picks one
  3. moves - takes game position to some other game position.
  4. goals - game position w/ a particular property

- assume optimal play
- dice rolls modelled by player playing randomly

Combinatorial Games

- Winning ways, On numbers and Games, Lessons in play
- usually 2-player
- Perfect information - every player knows the entire state of the game and moves available
- algebraic / combinatorial structure
- surreal #'s
- Algorithmic combinatorial game theory - better strategies than brute force

- Examples - games from lab

→ names chosen in honor of Richard Guy and his wife Louise. One of the founders of modern comb. game theory

- Def'n: A combinatorial game is a 2-player game played between Louise (for <u>Left</u>) and Richard (for <u>Right</u>). The game consists of the following:
  1. A set of possible positions, the states of the game.
  2. A move rule indicating for each positions what positions Louise can move to and what positions Richard can move to.
  3. A win rule indicating a set of terminal positions where the game ends. Each terminal position has an associated outcome - either Louise wins and Richard loses (denoted $+-$), Louise loses and Richard wins ($-+$), or it is a draw (00).

- To play - choose starting position and designate a player to go first.
  - take turns until terminal position is reached.

- Normal play games - win rule is is last player to move, i.e., the loser is the first player that can not move.

Game Trees

- each branch models a choice for a player and terminal nodes indicate an outcome
- Example Game Tic → Tic-tac-toe on 1 × 3 board, 2 adjacent squares win

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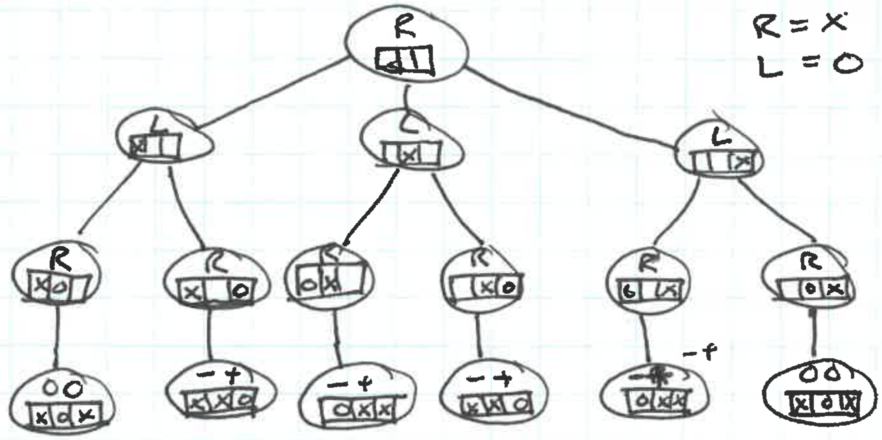


R = X
L = O

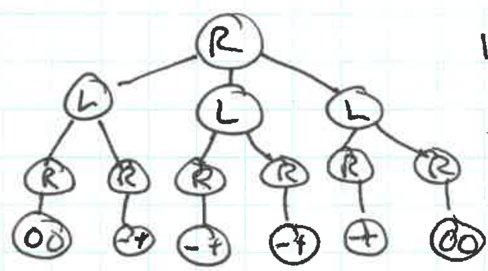
- each node contains the current position
- indicates whose turn it is
- terminal nodes indicate outcome R -+
  L +-
  OO Draw
- Start position – root

Build a Game Tree.

- starting at position α w/ Louise moving first
- Root node with L and α
- If L can move to $\alpha_1, \dots, \alpha_k$ then join k new nodes
  - each containing one of α_i position and R if nonterminal
  - If terminal, put outcome +-, -+, OO
- Repeat

WLD Trees.

- can just play game tree (choose which node)
- makes positional information superfluous
- Really only need whose turn it is and what the outcome is.



WLD game tree for Tic

- unified way to think about the play of combinatorial games

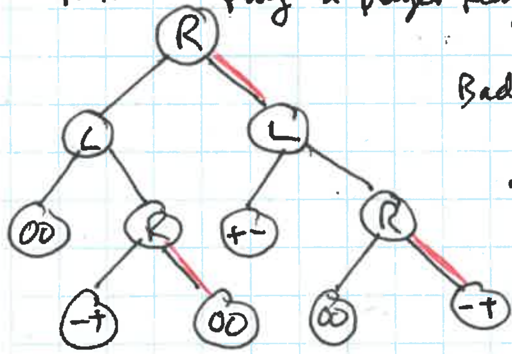
- Trivial trees

  -+ OO +-

- Trees could be ~~infinite~~ unbounded (chess). For now, limit ourselves to games w/ finite trees

Strategy (for a WLD tree)

- formalizes a plan to win
- a set of decisions indicating which move to make at each node where that player has a choice.
- winning strategy - best - guarantees a win
- drawing strategy - guarantees a player doesn't lose
- rational play - a player plays to win

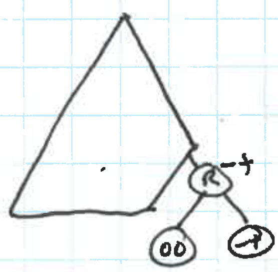


Bad strategy since L could choose to win

- Even though R always chooses to go right, we still show decisions in entire tree → makes strategies easier to work with

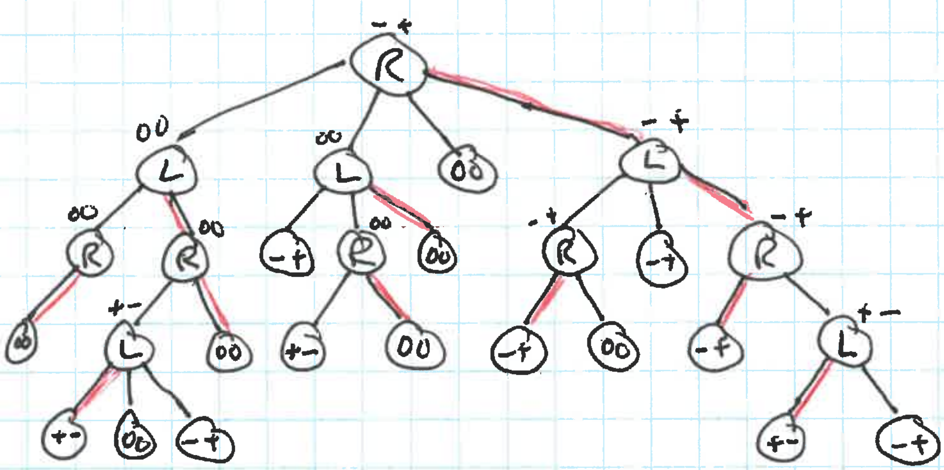
Working Backwards

- If we could consider entire tree at once it may be difficult to choose at the top, but not at the bottom



- since we know what decision R should make at this node, we can think of it as terminal and also mark it -+

- Procedure (working backwards): A player has a decision to make at node N. We've already determined the outcome under rational play for all possible nodes from node N. Choose a best possible outcome for this player. Indicate the choice in the graph and mark node N. Continue until root node is marked with an outcome



- Note under rational play Richard w/ always win b/c the root is marked -+

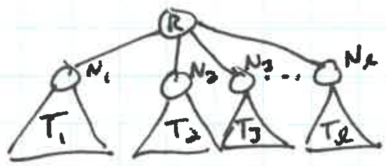
Zermelo's Thm.

- Every WLD game tree is one of

Type	Description
+ −	L has a winning strategy
− +	R has a winning strategy
0 0	Both players have drawing strategies

Proof. Via induction

- we know it works for trees that are size 1
- we know it works for trees up to i



cases: • At least one $T_1 \ldots T_k$ is type −+ choose

- All $T_1 \ldots T_k$ are type +− L wins
- None of $T_1 \ldots T_k$ are type −+, but ∃ one of type 00. pick it.

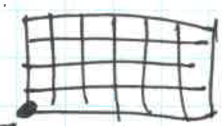
- Always get the Thm. from working backwards

☆

Strategy

- Symmetry - some games there is a symmetry
  - Prop. Consider an $m \times n$ position in chop
    (1) If $n = m$, the 2nd player has a winning strategy
    (2) If $n \neq m$, the 1st player has a winning strategy

Chop



☠ connected

$m \times n$ grid

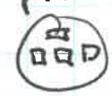
make horizontal or vertical cut

Proof sketch:

eventually only a 1×1 block (square) is left. So if you can keep the board as a square your opponent can only make a rectangle. Eventually, you win.

- Prop. Consider a pick-up-bricks position of n bricks.
  (1) If 3 divides n, the 2nd player has a winning strategy
  (2) Otherwise, the 1st player has a winning strategy.

☆ pick-up-bricks



- each player can pick up 1 or 2 bricks, last to move wins

Proof sketch. for (1), do the opposite of 1st player. Picks up 1, then pick up 2, etc. (3 every turn)
Eventually be left w/ last one or two bricks
For 2, pick up # to make n divisible by 3.

- Strategy stealing
  - prove existence of winning strategy w/o giving the strategy
  - proof by contradiction.

- Prop. The first player has a winning strategy in Hex (starting from an empty board)

Proofsketch. (by contradiction)

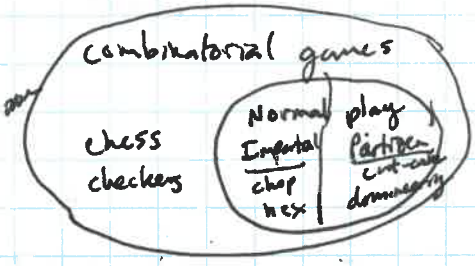
- Assume 2nd player has a winning strategy S on board $B = \{b_1, \dots, b_n\}$
- For 1st player move pick random spot b_i and place piece
  - Pretend you're 2nd player and use 2nd player strategy.
  - extra piece only benefits you.
- If strategy says to pick b_i, you already own it, so pretend you pick it and randomly pick another b_j
- Now you have 2nd player winning strategy $\Rightarrow \#$ since actually the 1st player.

$\Rightarrow$ 1st player has winning strategy.

- Note we don't know the strategy!

Normal play games

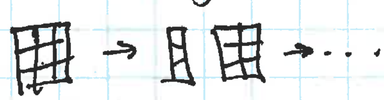
Venn diagram



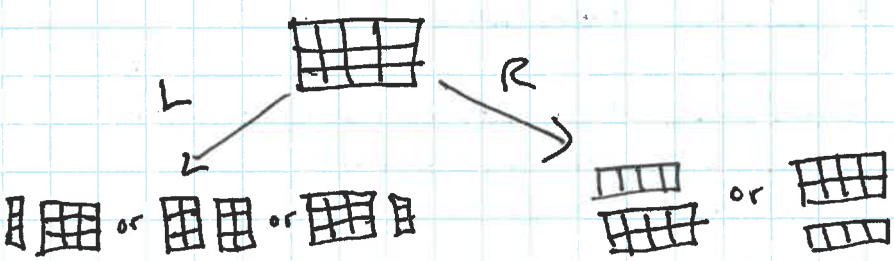
- Impartial - same moves are available to both players
- Partizan - different moves available to each player

Positions and types

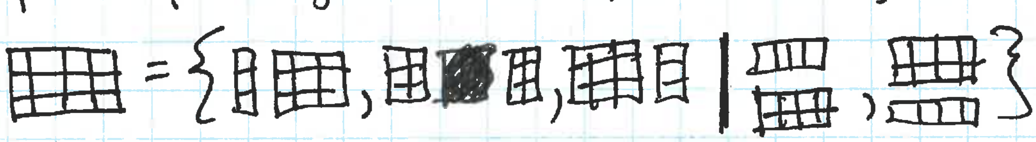
- Normal play games - set of positions w/ a rule dictating which positions L and R can move to.
- ✱ cut-cake
  - input $m \times n$ grid (cake).
  - L only makes vertical cuts, R makes horizontal cuts (in any piece)
  - last player that can make a cut wins



- Example position in cut-cake



- represent a position by moves of both $L, R \Rightarrow \{L \mid R\}$



- for simplicity we may only list different moves and ignore symmetric equivalent moves