# Combinatorial Games

Notes for Week (Sept. 2nd - Sept. 6th)

Team Q: Alissa Flores, Mauricio Flores, William Reckley, Roberto Rivas

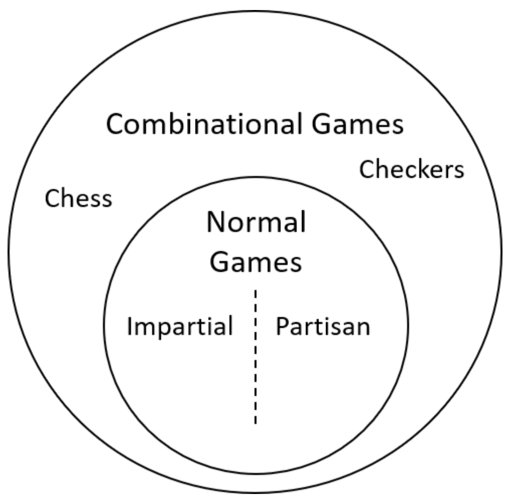
Notes from lab (8/30):

- 3D-Tic-Tac-Toe is a first player win. First player has a forcing strategy with getting the center cell of the board.
- Hex is a first player win. We can assume that first player doesn't have a winning strategy. If the second player has a winning strategy than first player can steal it and use it and with the extra move they made, they win.
- Dots and Boxes: There is an importance on the double cross strategy. This strategy allows you to sacrifice 2 boxes in order to a bigger chain of boxes.
    - For more information: How to always win at Dots and Boxes - Numberphile https://www.youtube.com/watch?v=KboGyIilP6k

## Normal Play Games

Definition: Where the winner is the last person to make a move

Examples: Pick Up Bricks, Chop, etc.




Impartial definition: The same moves are available to both players

Examples: Tic-Tac-Toe, Dots & Boxes

Partisan definition: Different moves available to each player

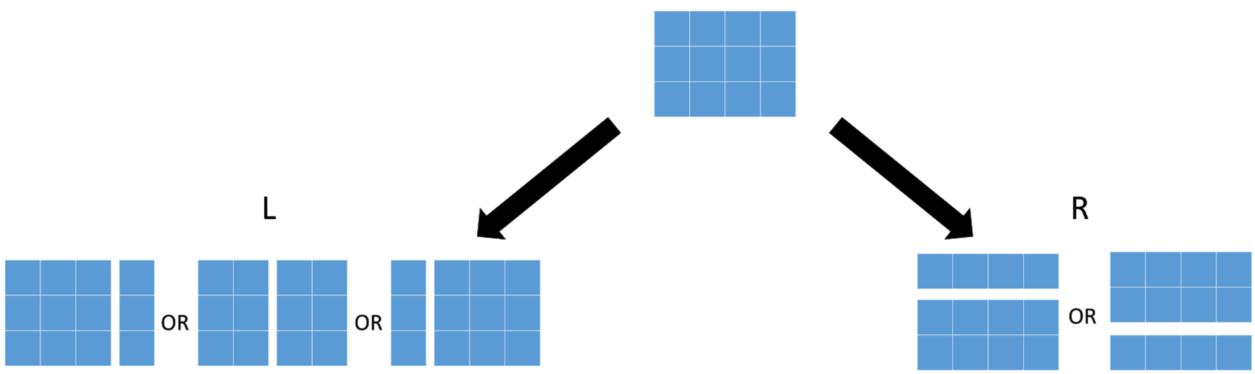
Example: Cut-Cake

# Positions and Types

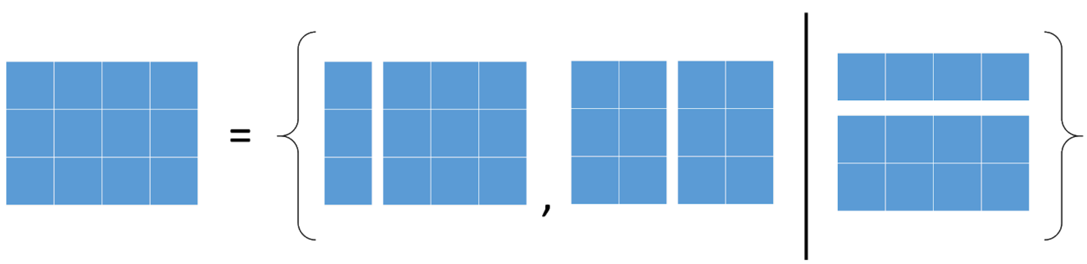
Normal Play games: set of position with a rule dictating which positions L & R can make.

**Cut-cake**

- mxn grid (cake)
- L only makes vertical cuts
- R only make horizontal cuts
- Last player that can make a move (cut) wins




Position = { L's moves | R's moves }



Abstractly $\boldsymbol{Y}$=$\{\boldsymbol{\alpha}_1,...,\boldsymbol{\alpha}_n|\boldsymbol{\beta}_1,...,\boldsymbol{\beta}_n\}$
For simplicity we ignore symmetric equivalent positions

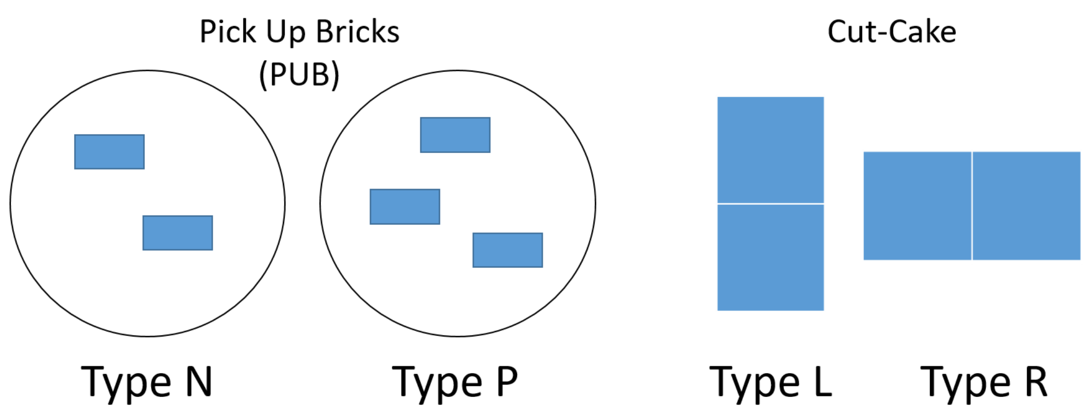
# Types of positions

**Zermelo's Theorem:** Sometimes L or R has a winning position, Sometimes 1st or 2nd has winning position.

**Correlation:** Every position in a normal play games is one of the following:

| Type | Description |
|---|---|
| L | Has a winning position regardless of who goes first |
| R | Has a winning position regardless of who goes first |
| N | The next player to go has a winning strategy |
| P | The previous to go has a winning strategy |

Examples:



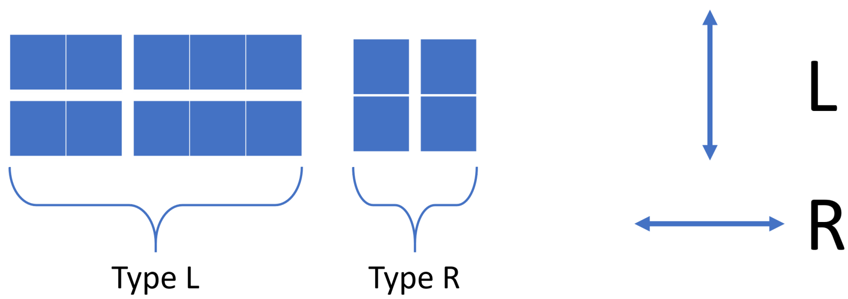
# Determining type

L moves first from position $\boldsymbol{\alpha}$ and has a move to $\boldsymbol{\beta}$ of type L or P, $\boldsymbol{\alpha}$ has a type N

**Proposition:** If $\gamma=\{\alpha_i,\ldots,\alpha_m \mid \beta_i,\ldots,\beta_n\}$, the type of $\gamma$ is given by the following

| | Some $\beta_i$ is type R or P | All $\beta_i, \ldots, \beta_n$ is type L or N |
|---|---|---|
| Some $\alpha_i$ is type L or P | N | L |
| All $\alpha_i, \ldots, \alpha_m$ is type R or N | R | P |

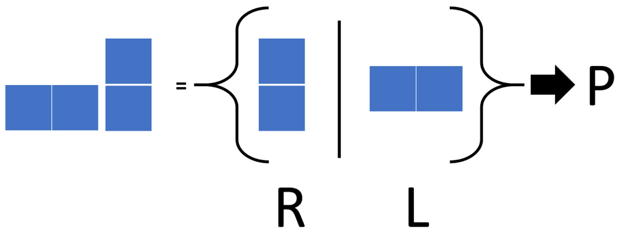
Examples:

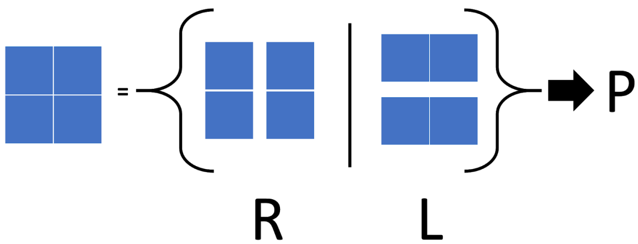


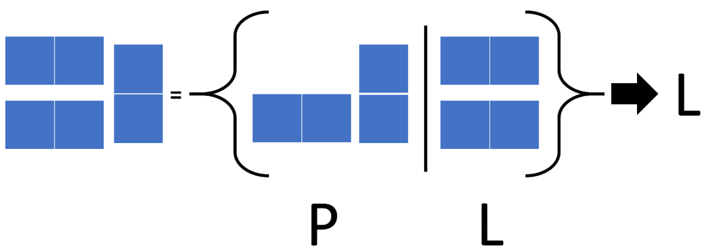

**Cut Cake:** The left player cuts vertically, and the right horizontally.

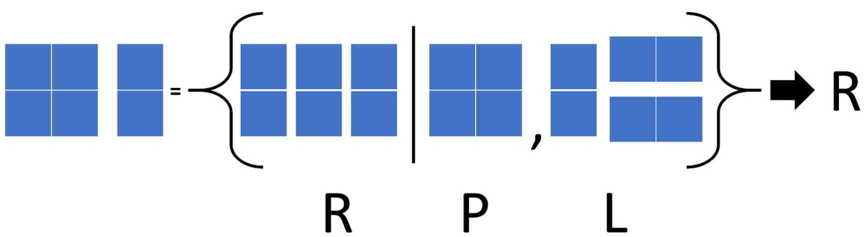
Given sample boards from Cut Cake and reviewing and labeling all moves that can be generated as N, R, L, and P we can then determine an overall type.

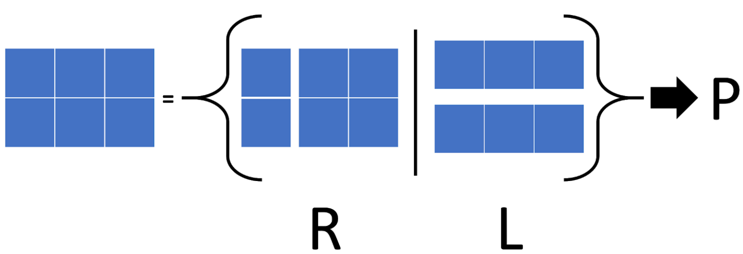
Examples:



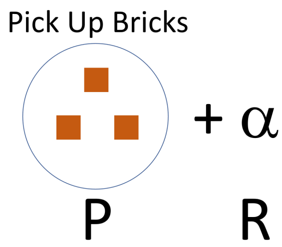



# Sums of Positions

Determine the position type based on the sum of its components.




Let $\alpha$ be some position in game of type R

What move should be made?

-Ignore P and play his winning strategy in $\alpha$

-If L moves in Pick Up Bricks (PUB) he responds in PUB

**Proposition:** If $\beta$ is type P, then $\alpha$ and $\alpha$+$\beta$ are the same type

**Proposition:** If both $\alpha$ and $\beta$ are type L(R) then $\alpha$+$\beta$ is type L(R)

Types of Sums

| + | L | R | N | P |
|---|---|---|---|---|
| **L** | L | ? | ? | L |
| **R** | ? | R | ? | R |
| **N** | ? | ? | ? | N |
| **P** | L | R | N | P |

? = game/position specific

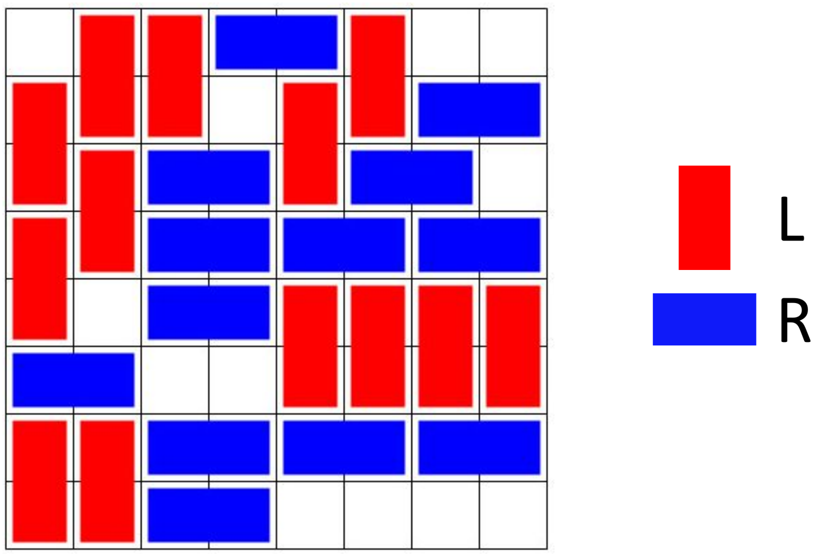
Determinate Sums - What we know (the letters)

Indeterminate Sums - What is unknown (the question marks)

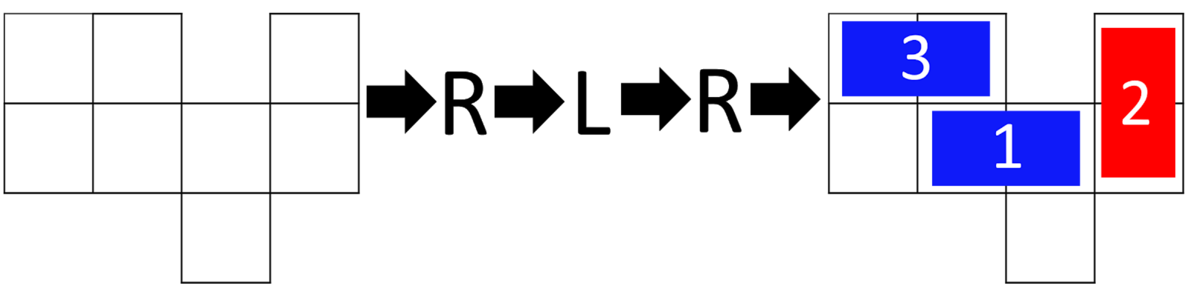
# Indeterminate Sums

**Domineering**

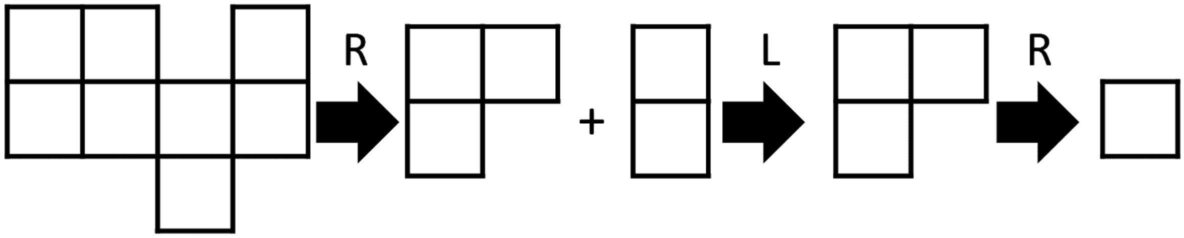
- 2 players in an m x n grid
- One player can only place vertical dominoes on the board while the other can only place horizontal dominoes
- Last player to make a move wins



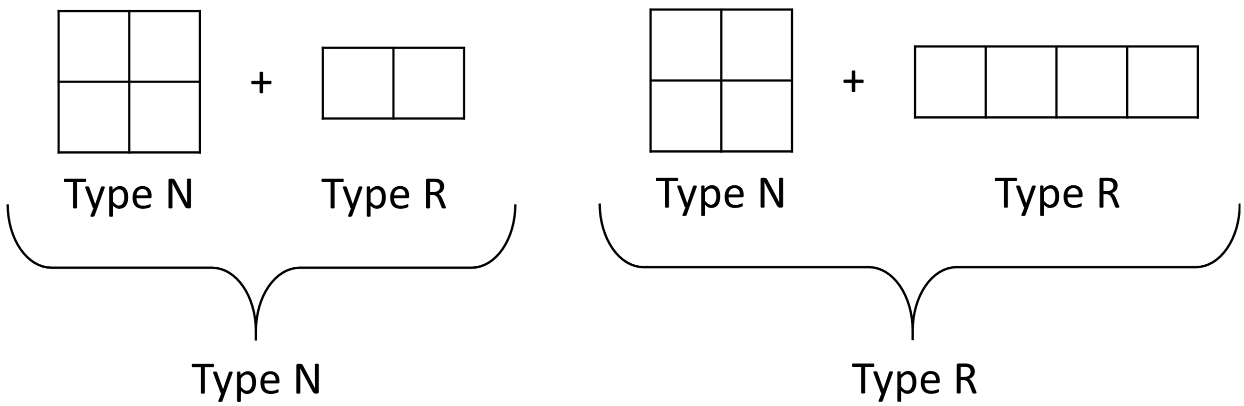

Example:




The figure above translates to the following:

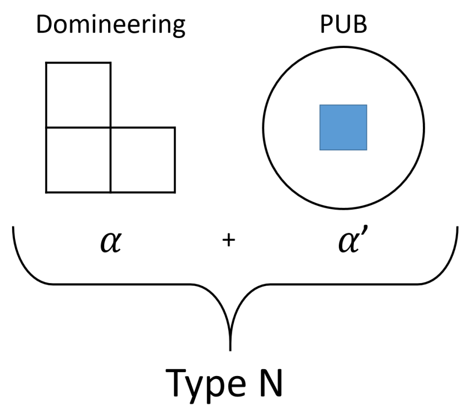


After viewing the previous, consider the following.




**Equivalence**

Definition: 2 positions $\alpha$,$\alpha$' in normal play games are equivalent if for every position $\beta$ in any NPG, the two positions $\alpha$+$\beta$ and $\alpha$'+$\beta$ have the same type.



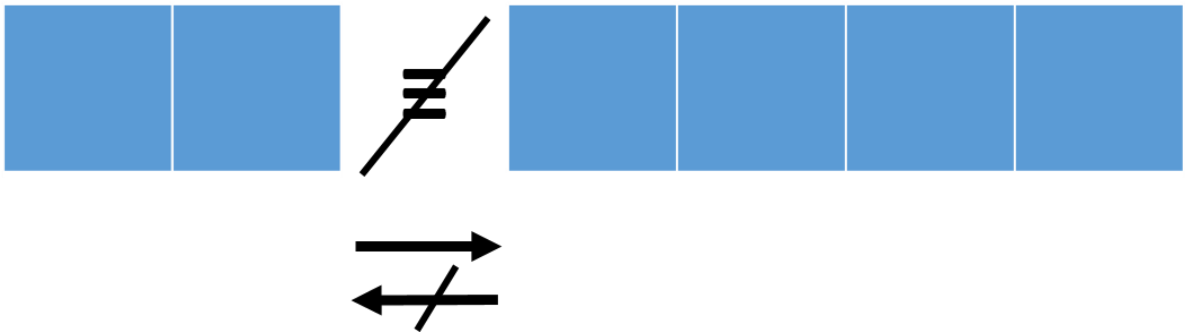

$$\alpha\text{'} \equiv \alpha$$

## Equivalence Relations

**Proposition:** If $\alpha$,$\beta$, $\gamma$ are position NPG, then

1. $\alpha \equiv \alpha$' (reflexivity)
2. $\alpha \equiv \beta \rightarrow \beta \equiv \alpha$ (symmetry)
3. $\alpha \equiv \beta$ and $\beta \equiv \gamma \rightarrow \alpha \equiv \beta$ (transitivity)

**Proposition:** If $\alpha \equiv \alpha'$, then they have the same type (this is only a one way relationship)



## <u>Algebra</u> (+, ≡)

**Proposition:** If $\alpha,\beta,\gamma$ are positions in normal play games, then

1. $\alpha+\beta \equiv \beta+\alpha$ (commutativity)
2. $(\alpha+\beta) + \gamma \equiv \alpha + (\beta + \gamma)$ (associativity)

**Lemma:** Given positions $\alpha,\beta$ in NPG

1. If $\alpha \equiv \alpha'$ then $\alpha+\beta \equiv \alpha'+\beta$
2. If $\alpha_i \equiv \alpha'_i$ for $1 \leq i \leq n$, then $\alpha_i +\ldots +\alpha_n \equiv \alpha'_i +\ldots +\alpha'_n$
3. If $\alpha_i \equiv \alpha'_i$ for $1 \leq i \leq n$ and $\beta_j \equiv \beta'_j$ for $1 \leq j \leq n$, then $\{\alpha_1,.., \alpha_n|\beta_1,...,\beta_j\} \equiv \{\alpha'_1,.., \alpha'_n|\beta'_1,...,\beta'_j\}$.

Type P in NPG behave like 0 under addition

**Lemma:** If $\beta$ is type, P, then $\alpha+\beta \equiv \alpha$

**Proposition:** If $\alpha$ and $\alpha'$ are type P, then $\alpha \equiv \alpha' \rightarrow \alpha+\gamma \equiv \gamma \equiv \alpha' + \gamma$

**Lemma:** If $\alpha + \beta$ and $\alpha' + \beta$ are both type P, then $\alpha \equiv \alpha'$

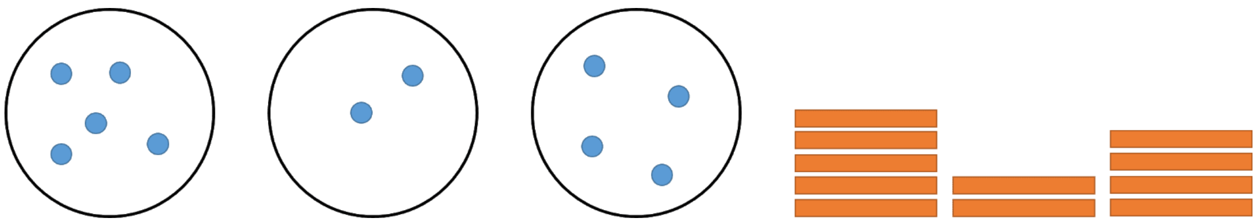
## <u>Impartial games</u>

Definition: all players have the same moves available

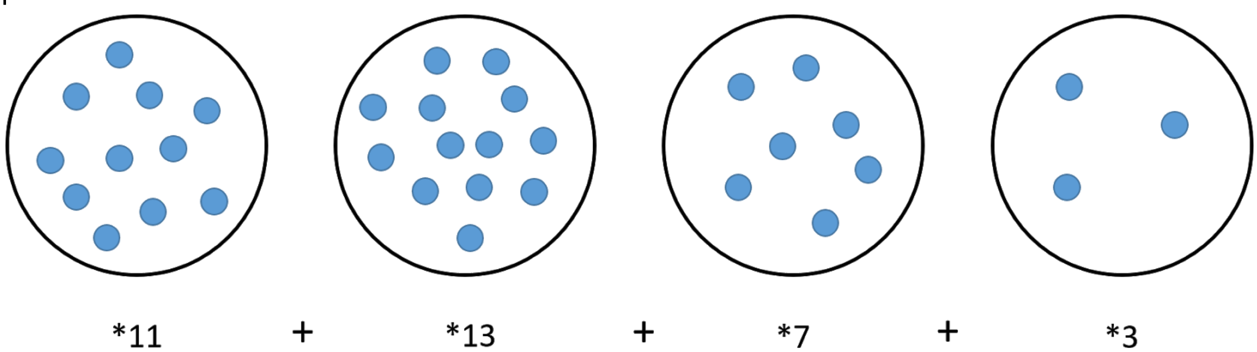
Every game position in an impartial game is either
Type N - there exist a move to a type p
(essentially a there exists)
Type P - there are no moves to a type p
(essentially a for all)

**Nim**

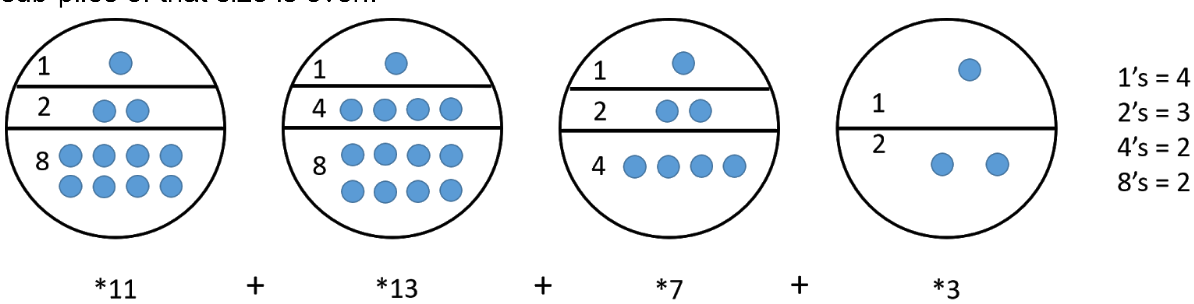
- an impartial game, where a position consists of x piles of stones of sizes $a_1, \dots, a_x$
- To move a player removes 1 to $a_i$ stones from stack i



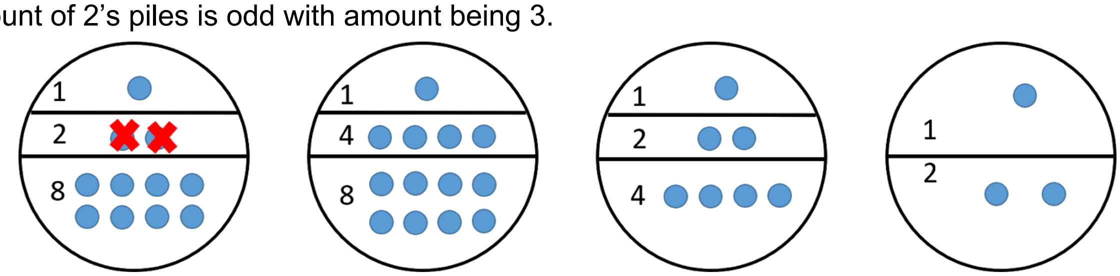
For ease of notation: use *a to denote a non negative integer a to denote that many stones in a pile.



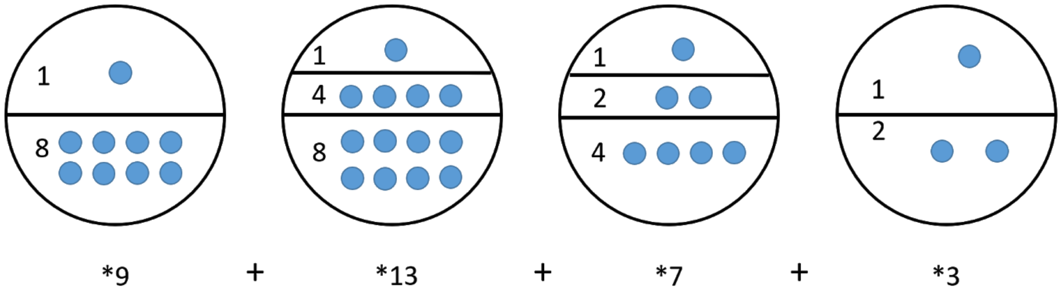

Given a Nim position *a + … *$a_k$, it is balanced if for every power of 2 the total number of sub-piles of that size is even.




After dividing, we must remove stones to make the amount sub-piles even. In this example the amount of 2's piles is odd with amount being 3.




After removing, it becomes <u>balanced.</u>




1's = 4
2's = 2
4's = 2
8's = 2