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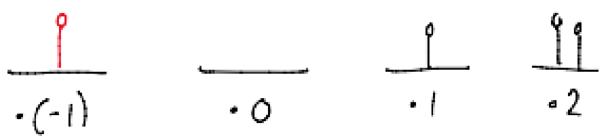
Games & Computation

September 25, 2019

HackenBush

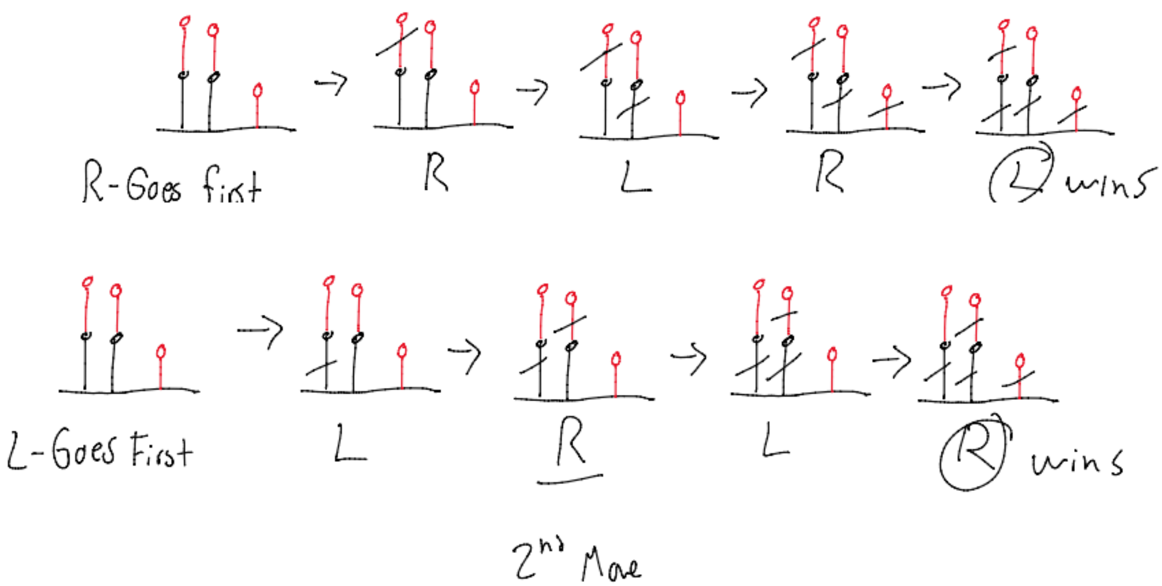


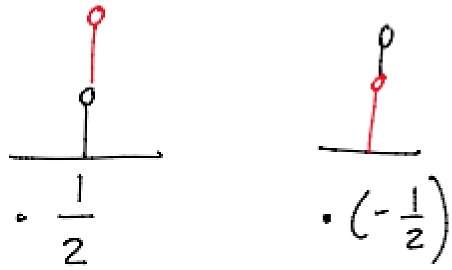
L = Black
R = Red

Integer positions

Fractional Positions

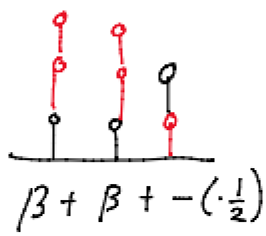
- Can we get these?
- An advantage at ½ : α + α = •1
- α + α + • (-1) ≡ • 0 ≡ P

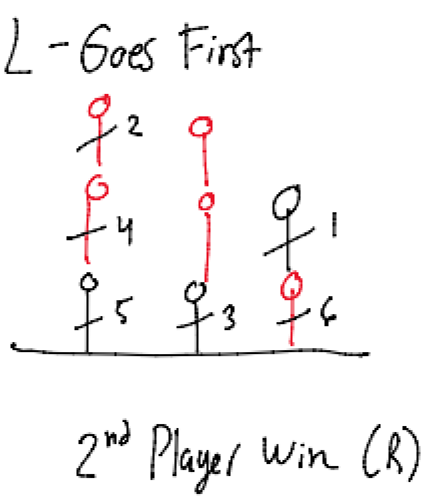


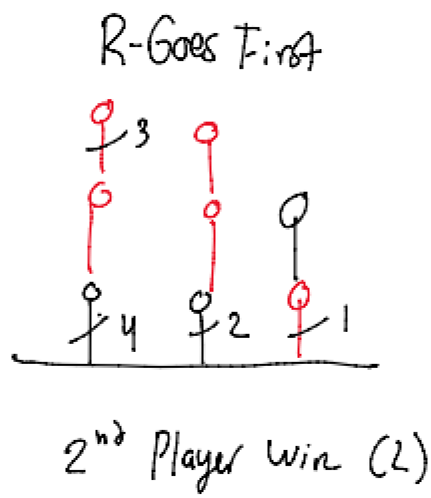


What about ¼ ?

If some ß ≡ • ¼ , then ß + ß (-α) ≡ • 0

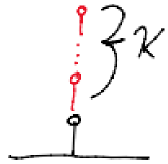






Familiarize These Positions

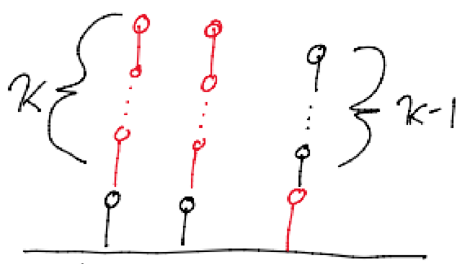
For every integer k * ($1/2^k$) is the HB position



$$\equiv \cdot\frac{1}{2^k}$$

Lemma: For every pos. int. k, we have:

$$\cdot\frac{1}{2^k} + \cdot\frac{1}{2^k} \equiv \cdot\frac{1}{2^{k-1}}$$



$$\cdot\frac{1}{2^k} + \cdot\frac{1}{2^k} + \left(-\cdot\frac{1}{2^{k-1}}\right)$$

Dyadic Numbers

- Any number that can be expressed as a fraction where the denominator is a power of 2 (and the numerator is an integer)
  - 17⁄16

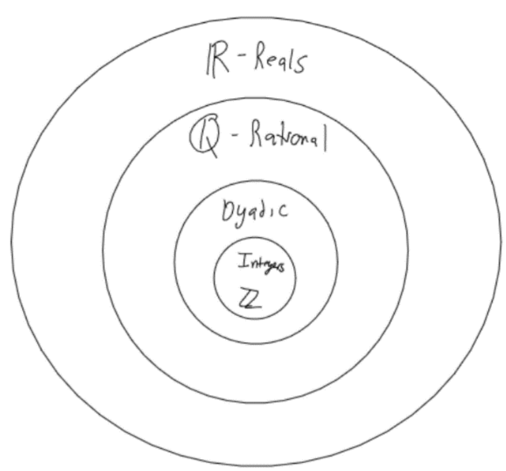
Proposition: Every dyadic number has a unique (finite) binary expansion

$83/64 = 1/64\ (83)$

$= 1/64\ (64 + 16 + 2 + 1)$

$= 1 + 1/4 + 1/32 + 1/64$

$= 1/2^0 + 1/2^2 + 1/2^5 + 1/2^6$



Birthdays

- Even though dyadic numbers are static, let's imagine generating them
- Each step/day generates new numbers

Day 0: 0

Day 1: $-1 \quad 0 \quad 1$

Day 2: $-2 \quad -1 \quad -\frac{1}{2} \quad 0 \quad \frac{1}{2} \quad 1 \quad 2$

Day 3: $-3 \quad -2 \quad -\frac{3}{2} \quad -1 \quad -\frac{3}{4} \quad -\frac{1}{2} \quad -\frac{1}{4} \quad 0 \quad \frac{1}{4} \quad \frac{1}{2} \quad \frac{3}{4} \quad 1 \quad \frac{3}{2} \quad 2 \quad 3$

General Procedure: On Day 0, the number 0 is born. If $a_1 < a_2 < a_3 \ldots < a_l$

are the numbers born on days 0, … , n, then on day n+1, the following numbers are born:

- Largest integer less than a_1
- Smallest integer greater than a_l
- The number $(a_i + a_{i+1})/2$ for every $1 \le i \le l - 1$

Proposition Every open interval of real numbers (a, b), (a, ∞), (-∞, b), or even (-∞, ∞) contains a unique oldest dyadic #'s. If we continue this to ∞, we get surreal #'s.

Define Positions

$\gamma = \{ \alpha, \ldots, \alpha m \mid B_1, \ldots, Bn \}$

Thus :

0 = { 1 } ≡ • o and *o ≡ Type P.

We get numbers based on using numbers from previous day.

$1 = \{0 \mid \}$, $-1 \{ \mid 0 \}$, $*1 = \{0 \mid 0\}$ // $\gamma = \{\alpha \mid \beta\}$

Left move — Right move — Both have same move

Next day:

$-2 = \{ \mid -1, 0 \}$, $2 = \{-1, 0, 1 \mid \}$ ... $\frac{1}{2} = \{-1, 0 \mid 1\}$, $-\frac{1}{2} = \{-1 \mid 0, 1\}$

$-2 = \{ \mid -1\}$, $2 = \{1 \mid \}$

$\{-1 \mid 1\} = 0$

$*2 = \{0, *1 \mid 0, *1\}$

$\uparrow = \{0 \mid *1\}$ // UP position

$\downarrow = \{*1 \mid 0\}$ // Down position

We can construct these to ∞.

If w is the simplest infinite number, then the simplest infinitesimal number is 1/w.

w * 1/w = 1

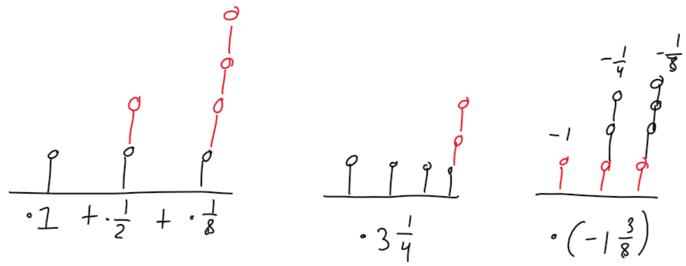
Back to the Party

- Dyadic positions – HB positions associated w/ dyadic numbers
- Every dyadic number, q>0, with binary expression $2^{d1} + 2^{d2} + \ldots + 2^{dl}$ where d1 > d2 > … > dl are integers (+/-)

Define position $\cdot q = \cdot 2^{d_1} + \cdot 2^{d_2} + \cdots + \cdot 2^{d_\ell}$

— R-red

+ L-black



$\cdot(-q) \equiv -(\cdot q)$

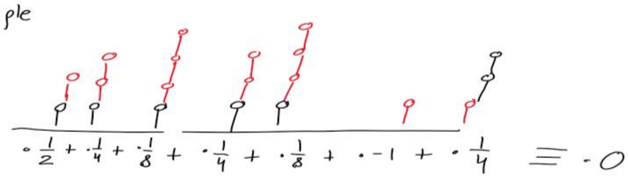
$\bullet(-q) \equiv -(\bullet q)$

Lemma. Let $a_1, \ldots, a_n$ be #s, each being 0 or of the form $(+/-)2^k$ for $k \in \mathbb{Z}$

If $a_1 + \ldots + a_n = 0$, then

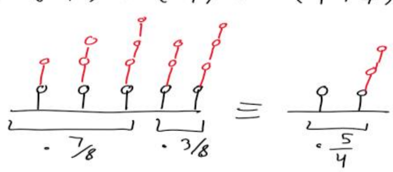
$\bullet a_1 + \ldots + \bullet a_n \equiv \bullet 0$

Example



Thm. If p, q are dyadic #'s, then

1. $-(\cdot p) \equiv \cdot(-p)$
2. $(\cdot p) + (\cdot q) \equiv \cdot(p+q)$



Obs. For a dyadic # q,

q is type $\begin{cases} L & \text{if } q > 0 \\ P & \text{if } q = 0 \\ R & \text{if } q < 0 \end{cases}$

Theorem: If p,q are dyadic numbers, then:

1. $-(\bullet p) \equiv \bullet(-p)$
2. $(\bullet p) + (\bullet q) \equiv \bullet(p+q)$

Observation: Fore a dyadic #q,
q is type:
L if q>0
P if q =0
R if q<0