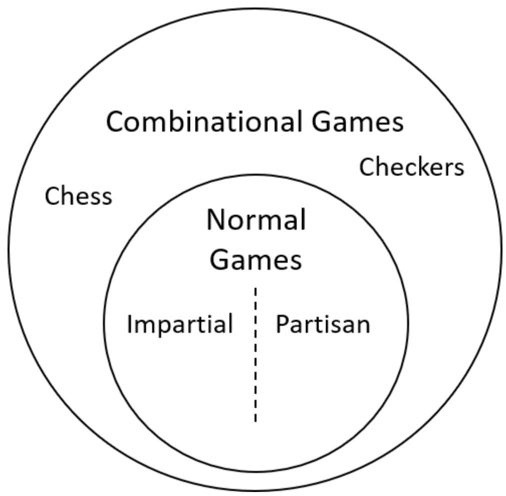
Notes for Week (1/23-1/27): Combinatorial Games

What did we learn from Friday's lab

- 3D-Tic-Tac-Toe is a first player win. The strategy is to simply by taking the center cell.
- Hex is a first player win.
- Dots and Boxes: There is an importance on the double cross strategy

Normal Play Games

Definition: Where the winner is the last person to make a move, ex: pick up brick, chop, etc.



Impartial: same moves are available to both players, ex: tic-tac-toe, dots & boxes

Partisan: different moves available to each player, ex: cut-cake

Position and Types

Normal Play games: set of position with a rule dictating which positions L & R can make.

Cut-cake

Input: mxn grid

L only makes vertical cuts and R only make horizontal cuts

Last player that can make a move wins

Images of cup cake

$\boldsymbol{Y}=\{\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_n|\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_n\}$

Types of positions

Zermelo's Theorem: Sometimes L or R has a winning position, Sometimes 1st or 2nd has winning position.

Type	Description
L	Has a winning position regardless of who goes first
R	Has a winning position regardless of who goes first
N	The next player to go has a winning strategy
P	The previous to go has a winning strategy

Examples of these types

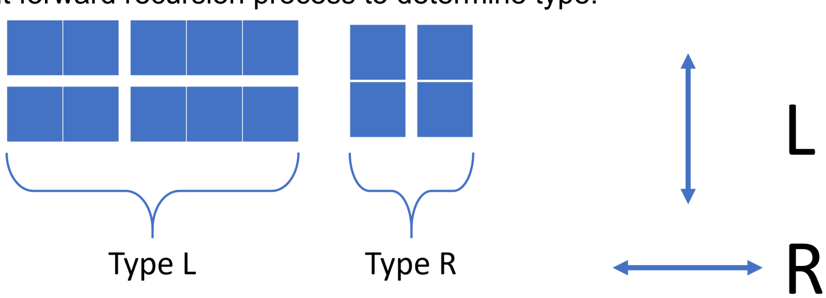
Determining type

L moves first from position α and has a move to β of type L or P, α has a type N

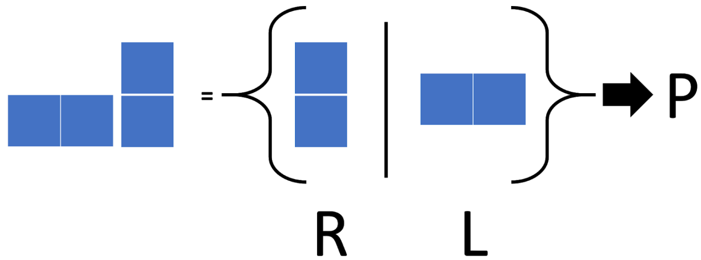
<u>Proposition:</u> if $\gamma=\{\alpha_i,\ldots,\alpha_m \mid \beta_i,\ldots,\beta_n\}$, the type of γ is given by the following

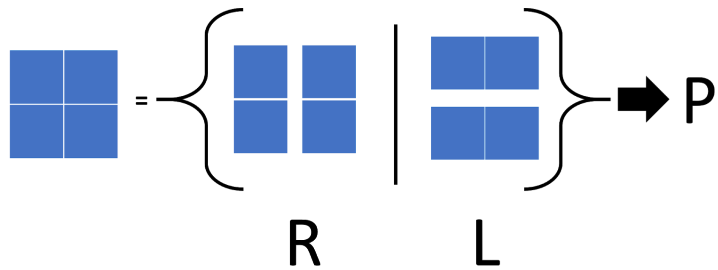
	Some β_i is type R or P	All $\beta_i, \ldots, \beta_n$ is type L or N
Some α_i is type L or P	N	L
All $\alpha_i, \ldots, \alpha_m$ is type R or N	R	P

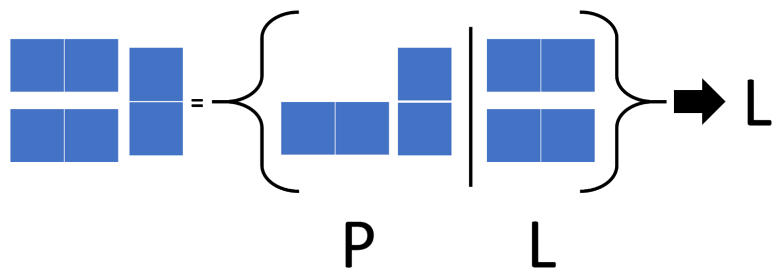
Straight forward recursion process to determine type:

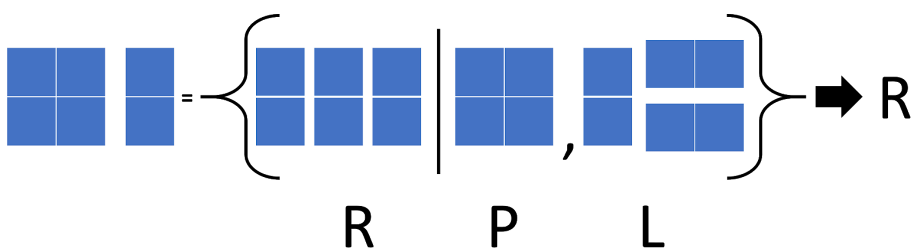


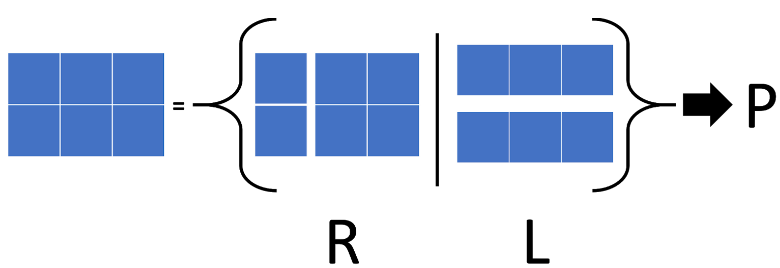
Examples:







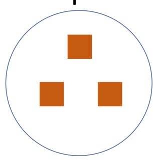




Sums of Positions

Determine the position type based on the sum of its components.

Pick Up Bricks

 + α

P R

Let α be some position in game of type R

What move should be made?

- -Ignore P and play his winning strategy in α
- -If L moves in Pick Up Bricks (PUB) he responds in PUB

Prop. If β is type P, then α and α+β are the same type

Prop. If both α and β are type L(R) then α+β is type L(R)

Type of Sums

+	**L**	**R**	**N**	**P**
L	L	?	?	L
R	?	R	?	R
N	?	?	?	N
P	L	R	N	P

? = game/position specific

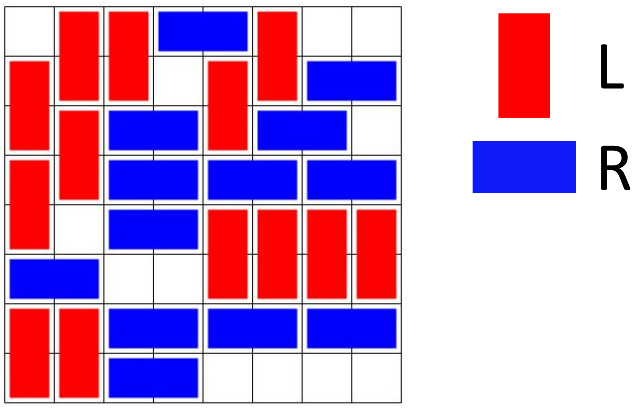
Determinate Sums - What we know (the letters)

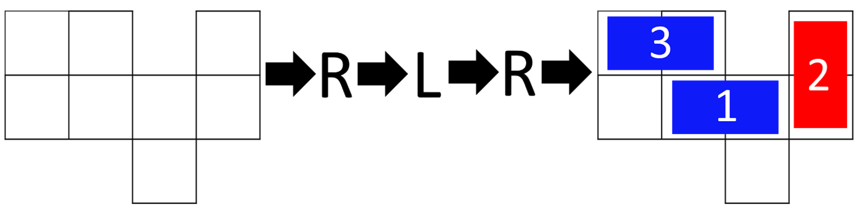
Indeterminate Sums - What is unknown (the question marks)

Indeterminate Sums

Example

Domineering: game played with 2 players in an m x n grid, were one player can only place vertical dominoes on the board while the other can only place horizontal dominoes. Last one to make a moves wins.





Equivalence

Definition: 2 positions α,α' in normal play games are equivalent if for every position β in any NPG, the two positions $\alpha+\beta$ and $\alpha'+\beta$ have the same type.

Equivalence relations

Prop. if α,β,γ are position NPG, then

1. $\alpha \equiv \alpha$ reflexivity
2. $\alpha \equiv \beta \rightarrow \beta \equiv \alpha$ Symmetry
3. $\alpha \equiv \beta$ and $\beta \equiv \gamma \rightarrow \alpha \equiv \beta$ transitivity

Prop. if $\alpha \equiv \alpha'$, then they have the same type (this is only a one way relationship)

Algebra with sums

Prop. if α,β,γ are positions in normal play games, then

1. $\alpha+\beta \equiv \beta+\alpha$ (commutativity)
2. $(\alpha+\beta) + \gamma \equiv \alpha + (\beta + \gamma)$ (associativity)

Lemma. Given positions α,β in NPG

1. If $\alpha \equiv \alpha'$ then $\alpha+\beta \equiv \alpha'+\beta$

2. If $\alpha_i \equiv \alpha'_i$ for $1 \leq i \leq n$, then $\alpha_i + ... + \alpha_n \equiv \alpha'_i + ... + \alpha'_n$
3. If $\alpha_i \equiv \alpha'_i$ for $1 \leq i \leq n$ and $\beta_j \equiv \beta'_j$ for $1 \leq j \leq n$, then $\{\alpha_1,.., \alpha_n|\beta_1,...,\beta_j\} \equiv \{\alpha'_1,.., \alpha'_n|\beta'_1,...,\beta'_j\}$.

Type P in NPG behave like 0 under addition

Lemma. If β is type, P, then $\alpha+\beta \equiv \alpha$
Prop. If α and α' are type P, then $\alpha \equiv \alpha' \rightarrow \alpha+\gamma \equiv \gamma \equiv \alpha' + \gamma$

Lemma. If $\alpha + \beta$ and $\alpha' + \beta$ are both type P, then $\alpha \equiv \alpha'$

Impartial games (all players have the same moves)
Every game position in an impartial game is either
Type N - there exist a move to a type p
    (essentially a there exists)
Type P - there are no moves to a type p
    (essentially a for all)

Nim - an impartial game, where a position consists of x piles of stones of sizes $a_1, \ldots, a_x$
    Move - a player removes 1 to a_i stones from stack i

For ease of notation: use *a to denote a non negative integer a to denote that many stones in a pile.
Given a Nim position $*a + \ldots *a_k$, it is balanced if for every power of 2 the total number of sub-piles of that size is even.