Impartial Games

Thursday, September 5, 2019 2:57 PM

Definition:

CG where both players have same moves available

Cor. Every pos, in an impartial game is type:

N - the next player has the winning strat

P - the previous player has the winning strat

Cor. A position in an impartial game is:

N - There exist a move to a type P pos

P - There are no moves to a type P pos.

Nim

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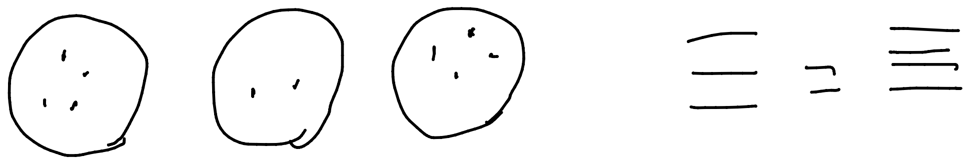
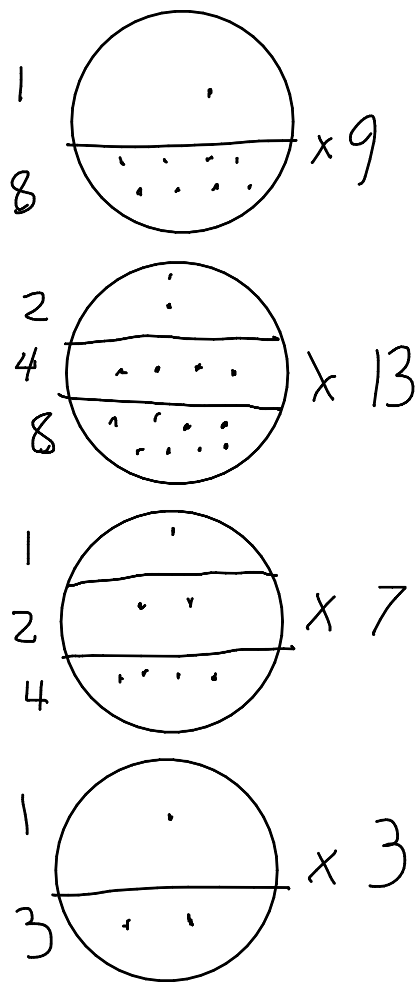
The Game of Nim is an impartial NPG where a pos consists of n piles of game pieces of sizes a_1, a_2, …, a_n.
Move - a player receives 1 to a_i, pieces from the stacks.



For ease of noation, *a to create a negative interger a to note that many pieces in that pile.

Def'n: Given a Nim position + a_1 + … + *a_k, for every power 2, the total # of sub-piles of that size is even



Nim

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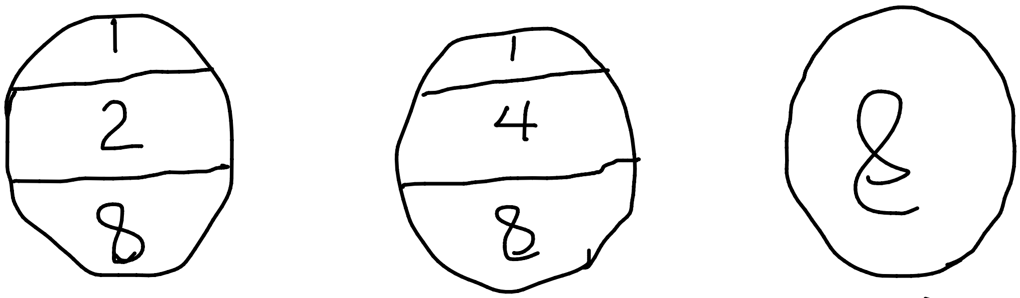

$*11$ $*13$ $*8$

$$2^k = 2^0 + 2^1 + \dots + 2^{k-1} + 1$$

Procedure-Balancing

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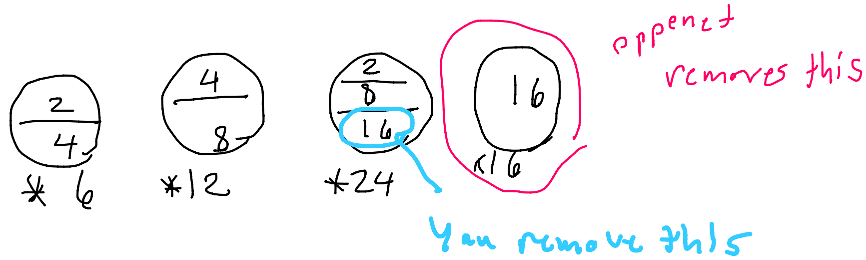
Let $*a_1+\ldots+*a_k$ be an unbalanced game of Nim.

:

2^M is the largest power of 2 with odd #s of subpiles. Chose a pile $*q_i$ which has a 2^M subpile. Currently all subpiles of 2^j w/ $j>m$ are even.

For every $j<m$, if odd # of subpiles of 2^j excluding this pile, leave 2^j stones.



Prop. Every balanced position of Nim is type P.
"unbalanced" is type N.

We call a Nim stack position a Nimber

$$P \equiv *0$$

DEF: The Nim-Sum of the non-ng int $a_1, \ldots, a_\ell$ denoted by $a_1 \oplus a_2 \oplus \ldots \oplus a_\ell$ is the non-neg int b w/ the property that 2^j appears in the binary expansion of b iff this term appears in an Odd # of times in the expansion of $a_1, \ldots, a_2$.

ex

or symbol

$$13 \oplus 19 \oplus 10 = (\cancel{8} + \underline{4} + \cancel{1}) \oplus (\underline{16} + \cancel{2} + \cancel{1}) \oplus (\cancel{8} + \cancel{2})$$
$$= 4 + 16$$
$$= 20$$

Thm. If $a_1, \ldots, a_\ell$ are non-neg int and $b = a_1 \oplus a_2 \ldots \oplus a_\ell$

then

$$*a_1 + \ldots + *a_\ell \equiv *b$$

Proof

$$(\underbrace{*a_1 + \ldots + *a_\ell + *b}_{\equiv *0}$$ is balance by defin. ~~of~~ b

$$\underbrace{*a_1 + \dots + *a_\ell}_{} \equiv *a_1 + \dots + *a_\ell + *0$$

$$\equiv *a_1 + \dots + a_\ell$$

$$\equiv *b$$

- Every Nim pos. is equal to a nimber
- Every impartial game corresponds to a nimber

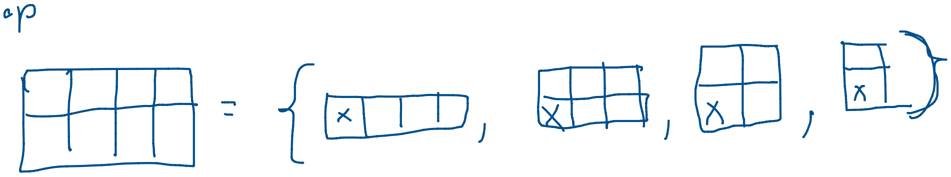
ex) Impartial NPG $\alpha = \{\beta_1, \dots \beta \mid \gamma_1, \dots, \gamma_m\}$

$\alpha = \{\alpha_1, \dots, \alpha_k\}$

Nim

$*4 = \{*0, *1, *2, *3\}$

Chop



Definition For a set $S = \{a_1, \dots, a_n\}$ of non-neg int, we define the Minimal Excluded Value of S to be the smallest non-neg int. b which is not one of $a_1, \dots, a_n$

ex) $\{0, 1, 2, 5, 8\}$ MEx 3

$\{1, 2, 3\}$ 0

$\{0, 1, 2, 3\}$ 4

Thm. (The MEX principle)

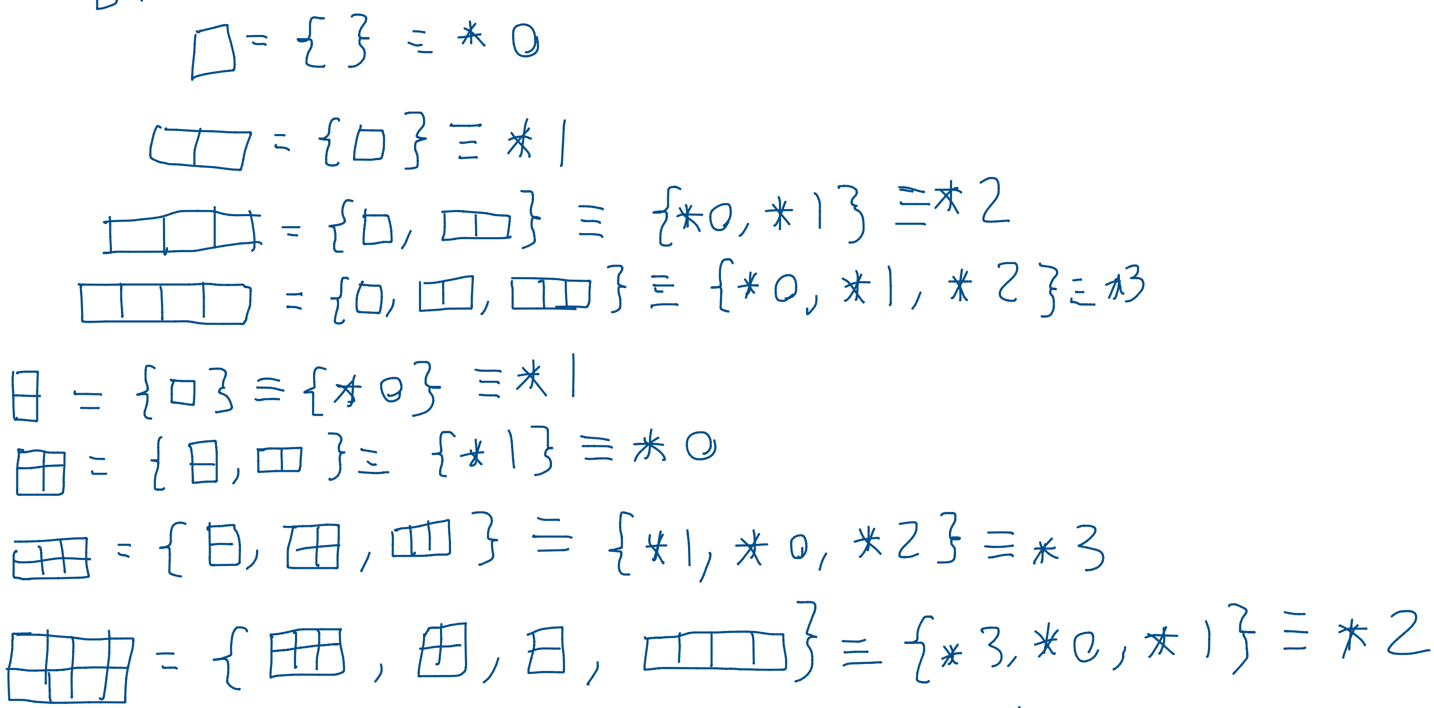
Let $\alpha = \{a_1, a_2, \ldots, a_k\}$ be a position in an impartial game

Suppose that $a_i \equiv * a_i$ for every $1 \le i \le k$

Then $\alpha \equiv *b$ where b is the MEX of the set $\{a_1, a_2, \ldots, a_k\}$

Thm (Sprague-Grundy)
Every position in an impartial game is equal to a nimber

Ex chop



Chomp

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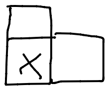
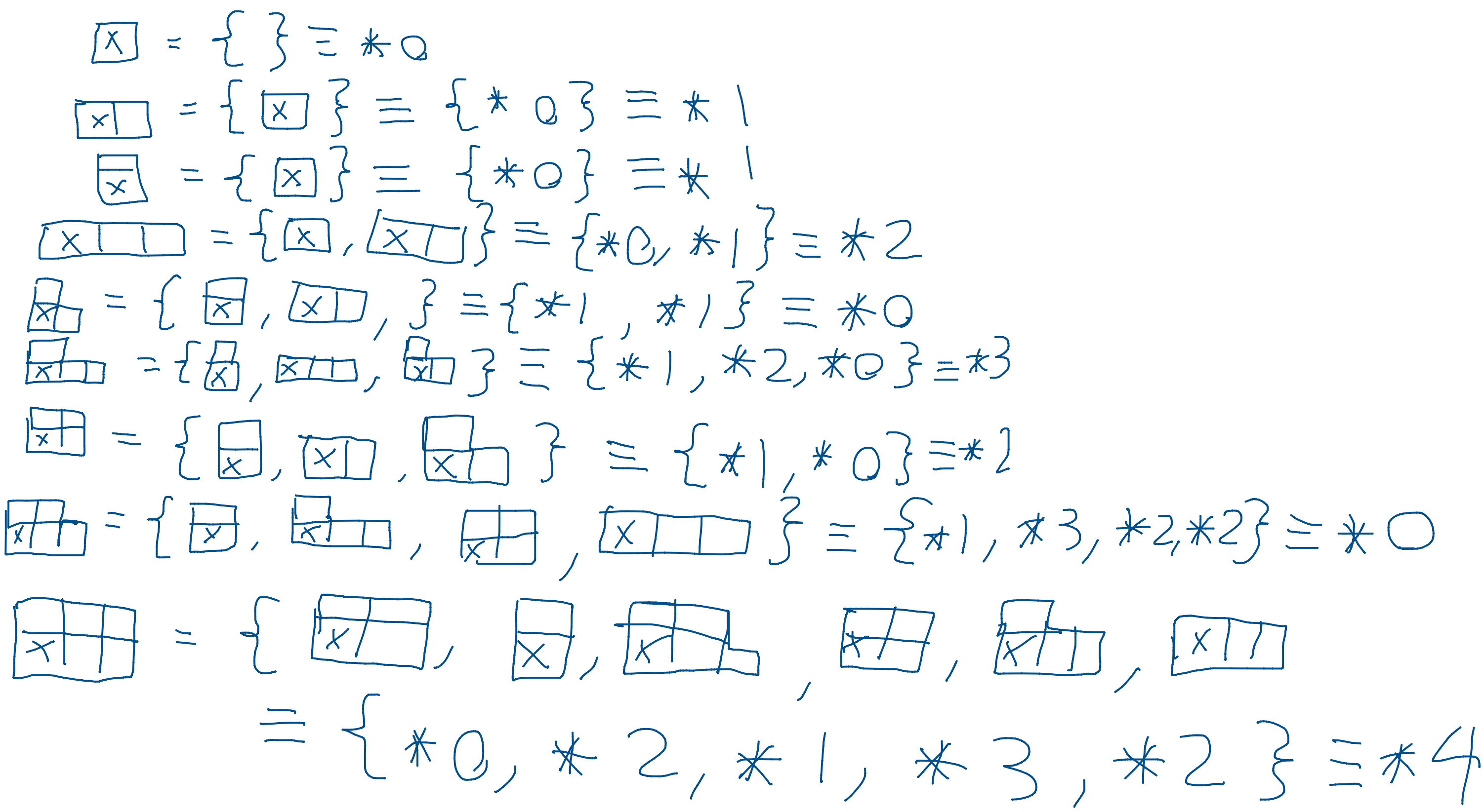
- input m×n
- pick a cell and n move everything
- Bottom left is anchor





Nimber equivalent to sums

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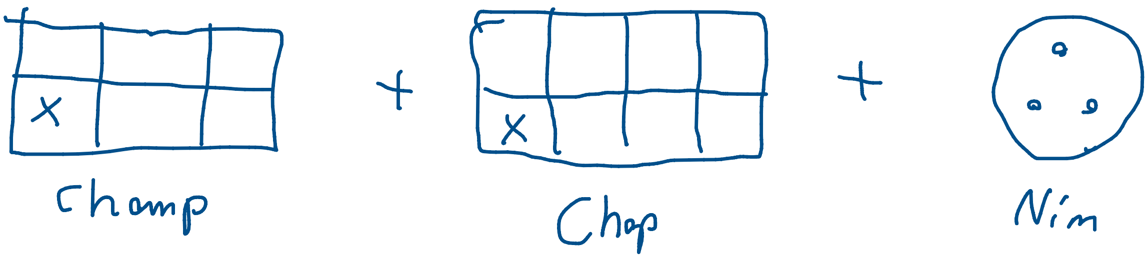
balanced



$$*4 + *2 + *3$$ not balanced

$$*(4 \oplus 2 \oplus 3)$$

$$*(4 \oplus \cancel{(2)} \oplus (\cancel{2}+1)) = *5$$

Pick up Bricks

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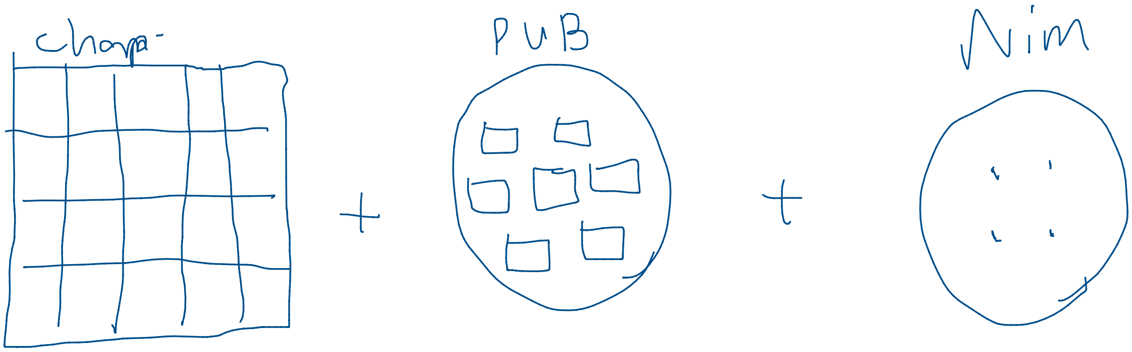
Theo. Let $n = 3\ell + k$ when $0 \le k \le 2$, then a (P.U.B) position is equal to $*k$

Chop:

w/ a hori. and vertical cut, an $m \times n$ chop board is equal to 2 NIM stacks, but leaving the last col/row

Theo. For every $m, n \ge 1$ an $m \times n$ position in Chop is equal to $*(m-1) + *(n-1)$



4x5

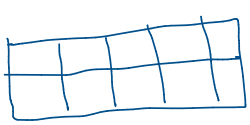
$$\underbrace{*7 \quad + \quad *1 \quad + \quad *4}_{*(7 \oplus 1 \oplus 4)}$$

$$*(3 \oplus 4) = *7$$

$$*(2 \oplus 4) = *6$$

$$*(1 \oplus 4) = *5$$

$$= *((\cancel{4} + 2 + \cancel{1}) \oplus \cancel{1} \oplus \cancel{4}) = *2$$



Partizan with Hack and Bush

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- for impartial $\Rightarrow$ every position equal to a nimber, which we compute w/ MEX

Hackenbush - NPG between L & R

The game board is a graph with colored edges of 2 colors, some of which are attached to the ground. A move for L is to cut/erase a-colored black edge, while R cuts a b-colored (red/gray) edge. Anything not connected to ground disappears

HB

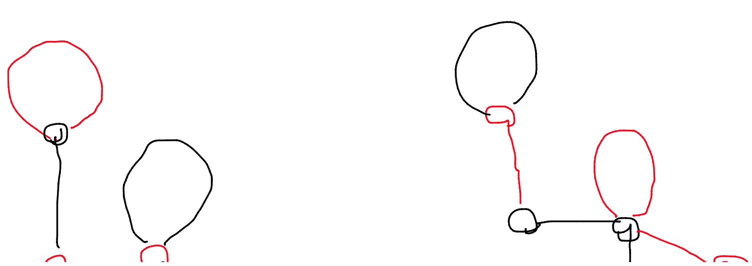
We want to represent positions by a number

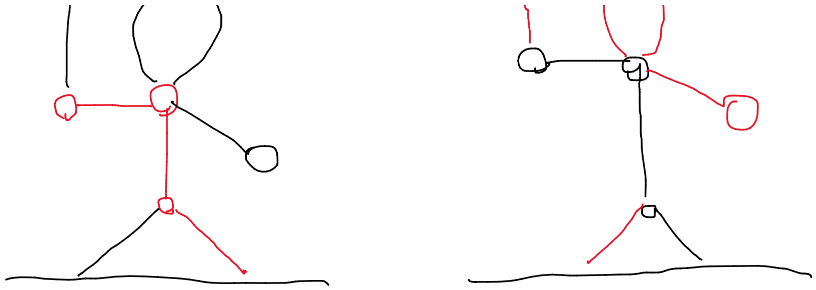
Defin.

- $\cdot 0$ is the HB position w/ no edge

Prop: A HB position α satisfies $\alpha \equiv \cdot 0$ iff α is type P

$$\Rightarrow \alpha + \cdot 0 \equiv \alpha$$





α $\qquad$ $-\alpha$ $\qquad$ (fig 1)

Negation

Reverse Colors $\longrightarrow$ reverses the colors of players

(fig 1)

Prop. If α and β are HB positions, then

1. $-(-\alpha) \equiv \alpha$
2. $\alpha + (-\alpha) \equiv \cdot 0$
3. $\beta + (-\alpha) \equiv \cdot 0$ implies $\alpha \equiv \beta$

Integer position

for $\forall$ position int n, define $\cdot n$ to be the HB position of n isolated black edges

$\ldots \quad \cdot(-2) \quad \cdot(-1) \quad \cdot 0 \quad \cdot 1 \quad \cdot 2 \quad \ldots$

negative changes color

Theo

For any int. m and n

1. $-(\cdot n) \equiv \cdot(-n)$
2. $(\cdot m) + (\cdot n) \equiv \cdot(m+n)$

1. $-(\cdot n) \equiv \cdot(-n)$

2. $(\cdot m) + (\cdot n) \equiv \cdot(m+n)$

for any $n > 0$, the intergal pos. $-n$ is type L and is an advantage for R

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Miljnieff - 2 player wins (P^{type})