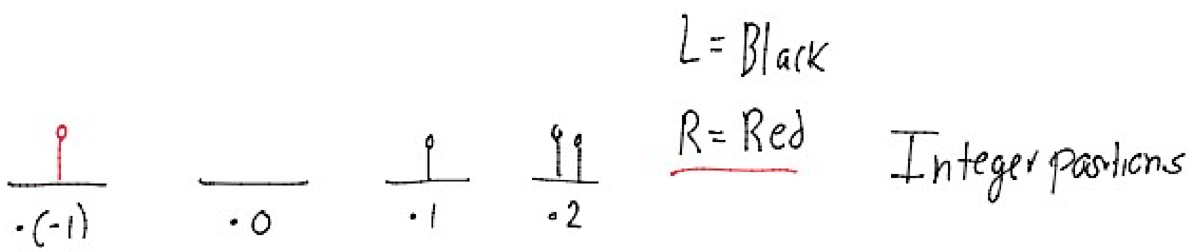
Jafet Reyes, Michael Alaniz, Salvador Jimenez, Eden Canales

Games & Computation

September 25, 2019

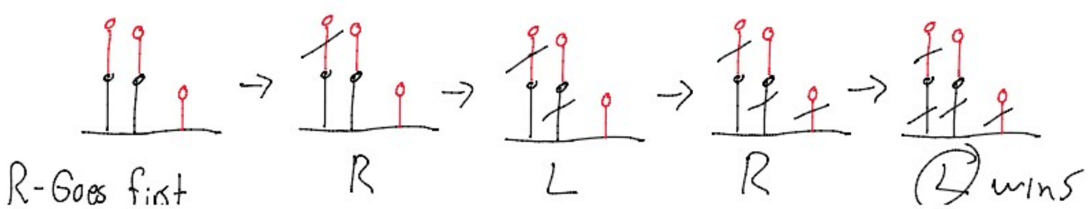
Day 7

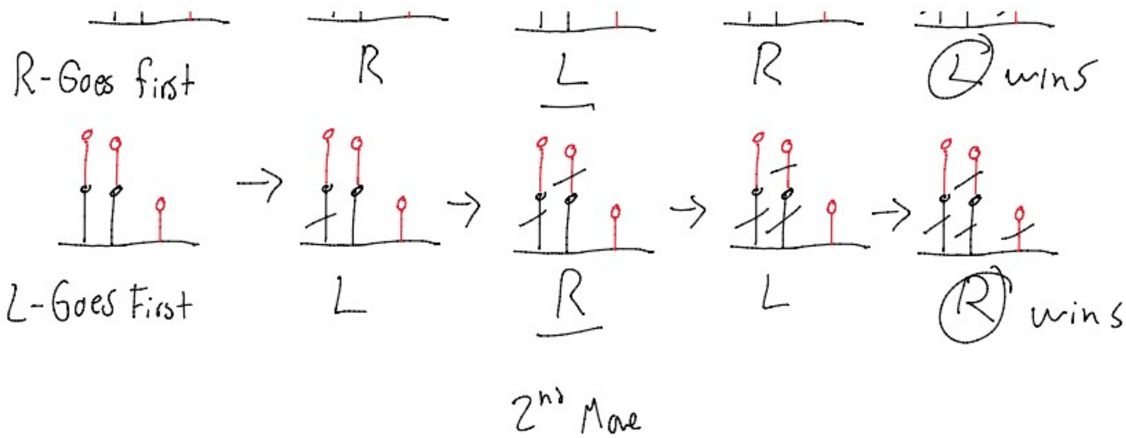
HackenBush

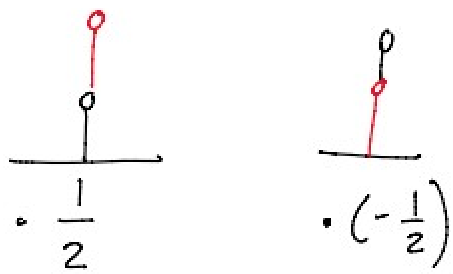


Fractional Positions

- Can we get these?
- An advantage at ½ : α + α = •1
- α + α + • (-1) ≡ • 0 ≡ P

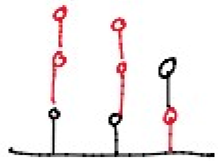






What about ¼ ?

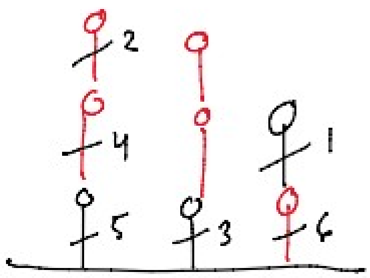
If some ß ≡ • ¼ , then ß + ß (-α) ≡ • 0

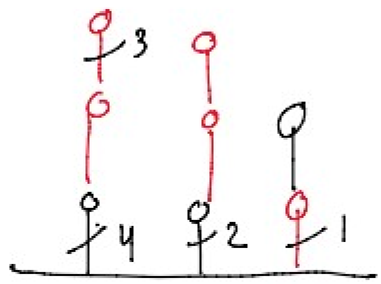


$\beta + \beta + -(\cdot\frac{1}{2})$

L-Goes First

R-Goes First



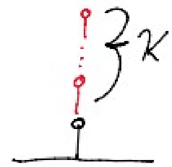


2^{nd} Player Win (R)

2^{nd} Player Win (L)

Familiarize These Positions

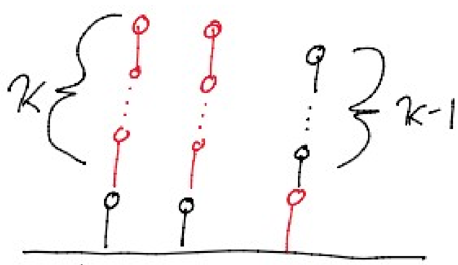
For every integer k $\cdot\frac{1}{2^k}$ is the HB position



$$\equiv \cdot\frac{1}{2^k}$$

Lemma: For every pos. int. k, we have:

$$\cdot\frac{1}{2^k} + \cdot\frac{1}{2^k} \equiv \cdot\frac{1}{2^{k-1}}$$



$$\cdot\frac{1}{2^k} + \cdot\frac{1}{2^k} + \left(-\cdot\frac{1}{2^{k-1}}\right)$$

Dyadic Numbers

- Any number that can be expressed as a fraction where the denominator is a power of 2 (and the numerator is an integer)
    - 17⁄16

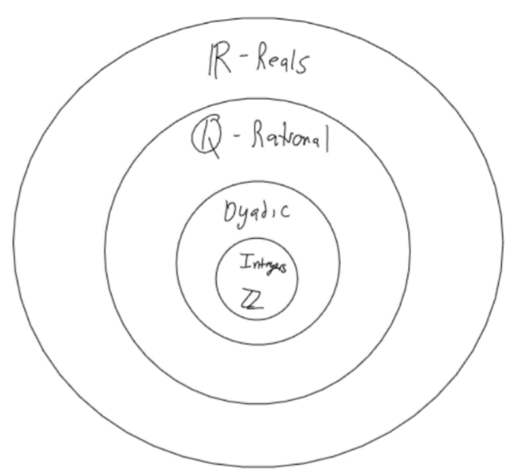
Proposition: Every dyadic number has a unique (finite) binary expansion

$83/64 = 1/64\,(83)$

$= 1/64\,(64 + 16 + 2 + 1)$

$= 1 + 1/4 + 1/32 + 1/64$

$= 1/2^0 + 1/2^2 + 1/2^5 + 1/2^6$



Day 8

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Birthdays

- Even though dyadic numbers are static, let's imagine generating them
- Each step/day generates new numbers

Day 0: 0

Day 1: $-1 \quad 0 \quad 1$

Day 2: $-2 \quad -1 \quad -\frac{1}{2} \quad 0 \quad \frac{1}{2} \quad 1 \quad 2$

Day 3: $-3 \quad -2 \quad -\frac{3}{2} \quad -1 \quad -\frac{3}{4} \quad -\frac{1}{2} \quad -\frac{1}{4} \quad 0 \quad \frac{1}{4} \quad \frac{1}{2} \quad \frac{3}{4} \quad 1 \quad \frac{3}{2} \quad 2 \quad 3$

General Procedure: On Day 0, the number 0 is born. If $a_1 < a_2 < a_3 \ldots < a_l$

are the numbers born on days 0, … , n, then on day n+1, the following numbers are born:

- Largest integer less than a_1
- Smallest integer greater than a_l
- The number $(a_i + a_{i+1})/2$ for every $1 \leq i \leq l - 1$

Proposition Every open interval of real numbers (a, b), (a, ∞), (-∞, b), or even (-∞, ∞) contains a unique oldest dyadic #'s. If we continue this to ∞, we get surreal #'s.

Define Positions

$\gamma = \{ \alpha, \ldots, \alpha m \mid B_1, \ldots, Bn \}$

Thus :

$0 = \{ 1 \} \equiv \bullet$ o and *o ≡ Type P.

We get numbers based on using numbers from previous day.

$1 = \{0 \mid \}$, $-1 \{ \mid 0 \}$, $*1 = \{0 \mid 0\}$ // $\gamma = \{\alpha \mid \beta\}$

Left move — Right move — Both have same move

Next day:

$-2 = \{ \mid -1, 0 \}$, $2 = \{-1, 0, 1 \mid \}$ ... $\frac{1}{2} = \{-1, 0 \mid 1\}$, $-\frac{1}{2} = \{-1 \mid 0, 1\}$

$-2 = \{ \mid -1 \}$, $2 = \{1 \mid \}$

$\{-1 \mid 1\} = 0$

$*2 = \{0, *1 \mid 0, *1\}$

$\uparrow = \{0 \mid *1\}$ // UP position

$\downarrow = \{*1 \mid 0\}$ // Down position

We can construct these to ∞.

if ω is the simplest infinite #, then simplest infinitesimal # is $\frac{1}{\omega}$.

$\omega \cdot \frac{1}{\omega} = 1$

Back to the Party

- Dyadic positions – HB positions associated w/ dyadic numbers

Every dyadic # $q > 0$ w/ binary expression $2^{d_1} + 2^{d_2} + \dots + 2^{d_\ell}$

where $d_1 > d_2 > \dots > d_\ell$ are integers (+/-)

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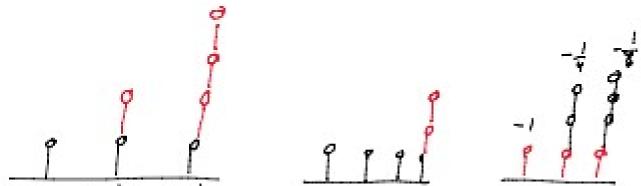
Define position $\cdot q = \cdot 2^{d_1} + \cdot 2^{d_2} + \cdots + \cdot 2^{d_\ell}$

$-$ R-red

$+$ L-black



$\cdot 1 + \cdot\frac{1}{2} + \cdot\frac{1}{8}$ $\qquad \cdot 3\frac{1}{4} \qquad \cdot\left(-1\frac{3}{8}\right)$

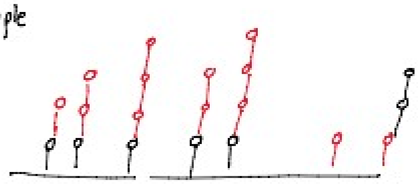
$\cdot 1\frac{5}{8}$

$\cdot(-q) \equiv -(\cdot q)$

Lemma. Let $a_1, \dots, a_n$ be #'s, each being 0 or of the form $\pm 2^k$ for $k \in \mathbb{Z}$

If $a_1 + \dots + a_n = 0$, then $\cdot a_1 + \dots + \cdot a_n \equiv \cdot 0$.

Example



$\cdot\frac{1}{2} + \cdot\frac{1}{4} + \cdot\frac{1}{8} + \cdot\frac{1}{4} + \cdot\frac{1}{8} + \cdot -1 + \cdot\frac{1}{4} \equiv \cdot 0$

Thm. If p, q are dyadic #'s, then

1. $-(\cdot p) \equiv \cdot(-p)$
2. $(\cdot p) + (\cdot q) \equiv \cdot(p+q)$

$\cdot\frac{7}{8} \quad \cdot\frac{3}{8} \quad \equiv \quad \cdot\frac{5}{4}$

Obs. For a dyadic # q,

q is type $\begin{cases} L & \text{if } q>0 \\ P & \text{if } q=0 \\ R & \text{if } q<0 \end{cases}$