CSCI-4341-01
Games & Computation
Week 3 Notes
Algebraic Combinatorial Game Theory

Sums of Positions

Definition: If α and β are positions in normal-play games, define α + β to be a new position consisting of components from α and β. To make a move a player picks which components to move in.

Given two normal-play game states and the ability to make a move in one of them each turn, transitioning the type of one game also transitions the type of the combination of both games. You can also break down certain combinatorial games into their individual components.

- Determinate Sums

  The heart of determinate sums is in proposition 2.6. and proposition 2.7. of "Game Theory: A Playful Introduction". Both propositions more or less describes how a type in β influences the type of α, α + β, and vice versa.

  As noted in the book, "our current state of knowledge concerning the behavior of types under summation is given by the table in Figure 2.11". I've included it here for your convenience

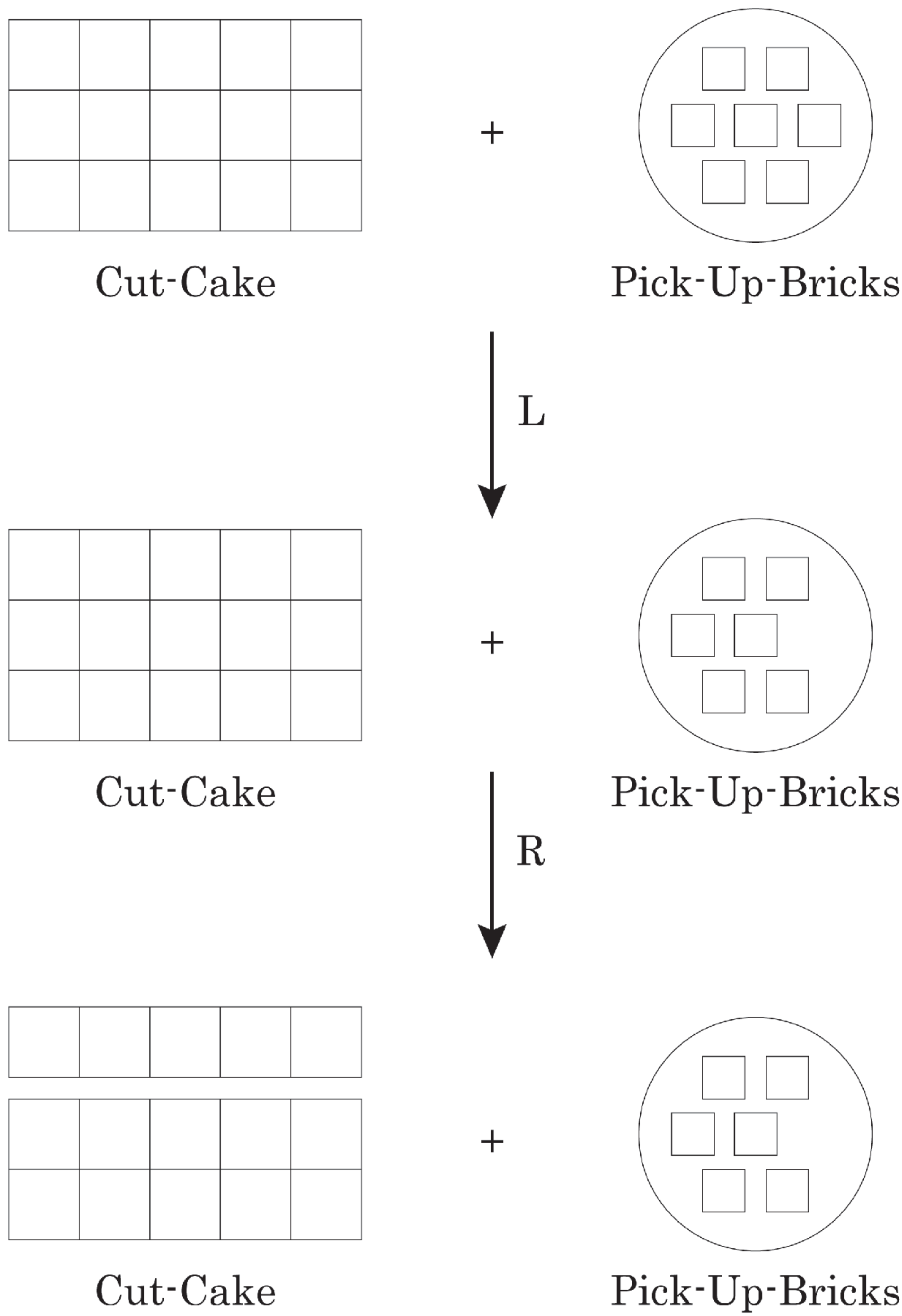
+	L	R	N	P
L	L	?	?	L
R	?	R	?	R
N	?	?	?	N
P	L	R	N	P

- Indeterminate Sums

A game that has an indefinite sum, is a game that can have a random number of components left but does not have a set sum due to the number of components left. The sum differs from the configuration of the game board at that random number of components. Common examples of this are heavily tied with the game domineering.

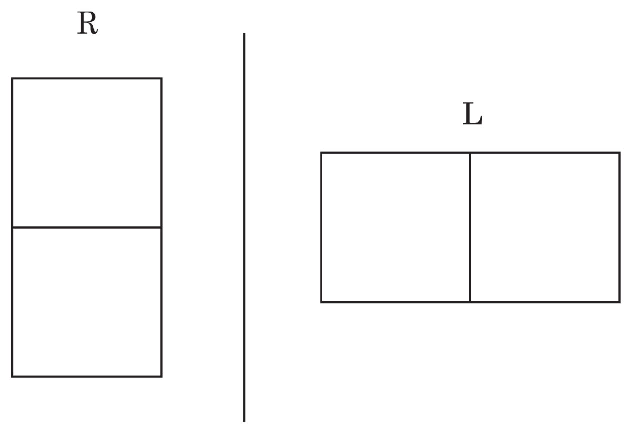
Domineering is a game where two players take turns taking up spaces on a grid. In domineering, the last player to move is the winner. Both players have different ways of moving. L can take up a 2 x 1 section of the grid and R can take up a 1 x 2 section of the grid.

Let's say it is towards the end of the game and there are 4 components left. There are different configurations for 4 components, but we will use a 2 x 2 grid and a 1 x 4 grid. The 2 x 2 grid is type N because the next player to make a move will win and the 1 x 4 grid is type R because only Richard has available moves. If it is Richard's turn to play, the 2 x 2 can be seen as N + R of type N. The 1 x 4 grid can be seen as N + R of type R. This shows that even though both grids had the same number of components left there was no way to determine the type of sum. Hence indefinite sum.

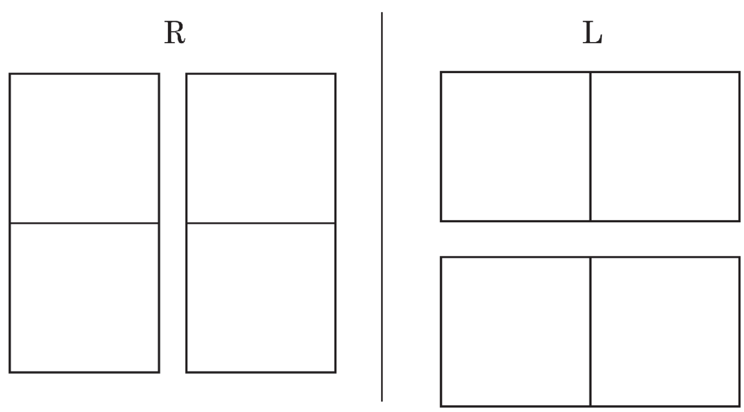

Cut-Cake
+
Pick-Up-Bricks
L
Cut-Cake
+
Pick-Up-Bricks
R
Cut-Cake
+
Pick-Up-Bricks

Determinate Sums Example

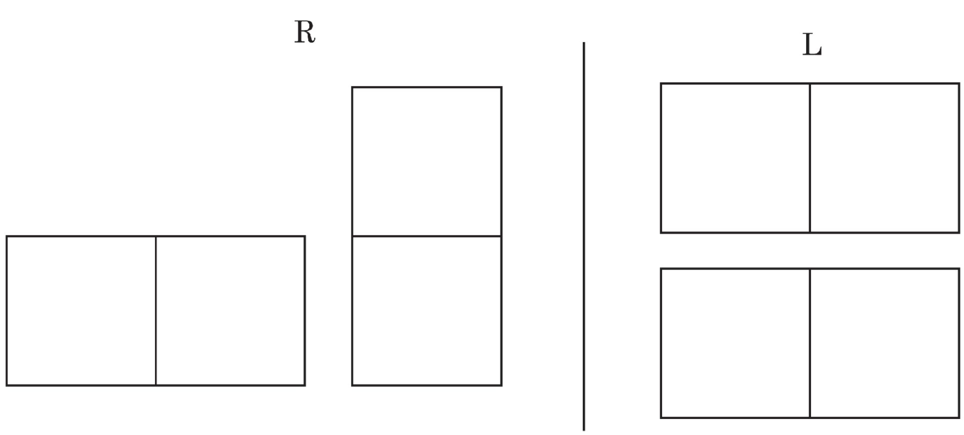
Type P



Type P

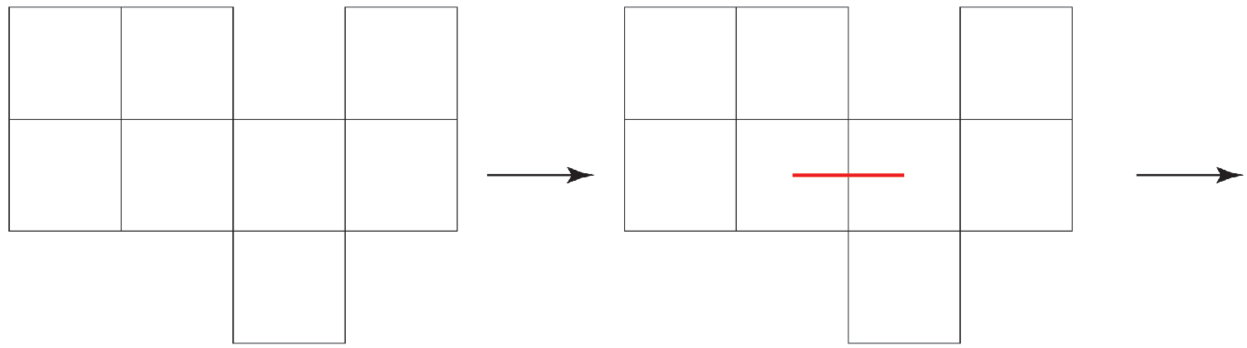


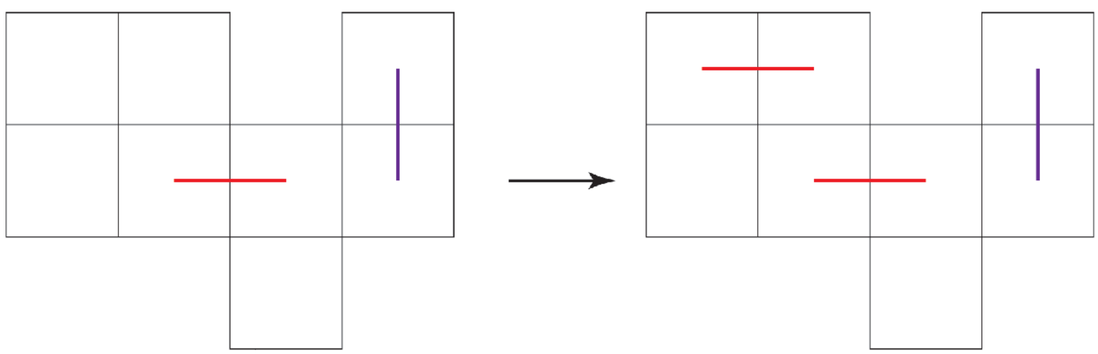
Type L

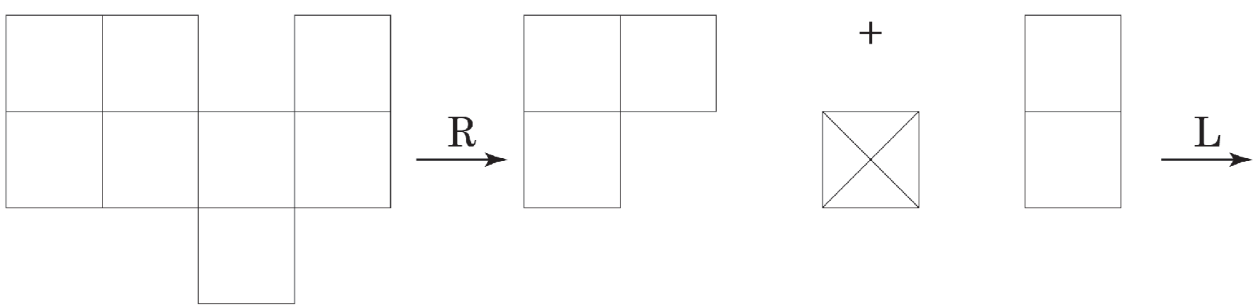


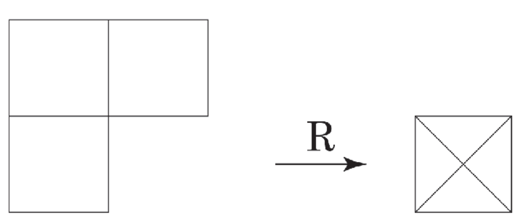
<u>Indeterminate Sums Domineering Example</u>

Assume R is horizontal, L is vertical

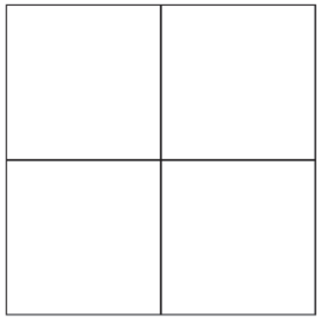




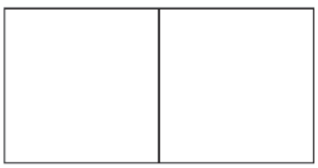




Type N



Type R



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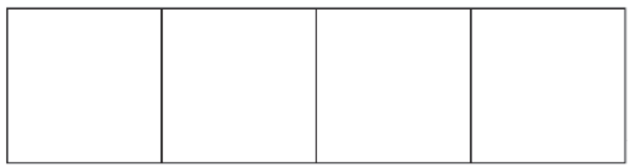
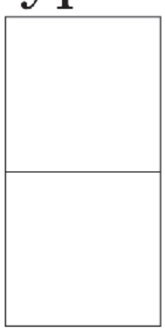
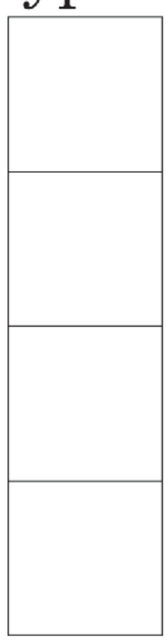
Type R



Type L



Type L



Equivalence

Definition: Two positions α and α' in normal play games are equivalent if all positions β in any normal-play game, $\alpha + \beta$ and $\alpha' + \beta$ have the same type.

Equivalence deals with multiple games at a time. Player N wins in either game. If we take any other game position β, then $\alpha + \beta \equiv \alpha' + \beta$ because play is equivalent.

Equivalence Relations

If α, β, and γ are positions in normal-play games

- $\alpha \equiv \alpha$ (Reflexivity)
- $\alpha \equiv \beta$ implies $\beta \equiv \alpha$ (Symmetry)
- If $\alpha \equiv \beta$ and $\beta \equiv \gamma$, then $\alpha \equiv \gamma$ (Transitivity)
- $\alpha + \beta \equiv \beta + \alpha$ (Commutativity)
- $(\alpha + \beta) + \gamma \equiv \alpha + (\beta + \gamma)$ (Associativity)

Given positions α, β in normal-play games

- If $\alpha \equiv \alpha'$, then $\alpha + \beta \equiv \alpha' + \beta$
- If $\alpha_i \equiv \alpha_i'$ for 1 <= i <= n, then $\alpha_i + . . . + \alpha_n \equiv \alpha_1 + . . . + \alpha_n$
- If $\alpha_i \equiv \alpha_i'$ for 1 <= i <= m and $\beta_j \equiv \beta_j'$ for 1 <= j <= n, then $\{\alpha_1, . . ., \alpha_m \mid \beta_1, . . ., \beta_n\} \equiv \{\alpha_i, . . ., \alpha_m' \mid \beta_i, . . ., \beta_n'\}$