CSCI 4341-01
Games and computation
Week 4 notes
Impartial and Partizan Games

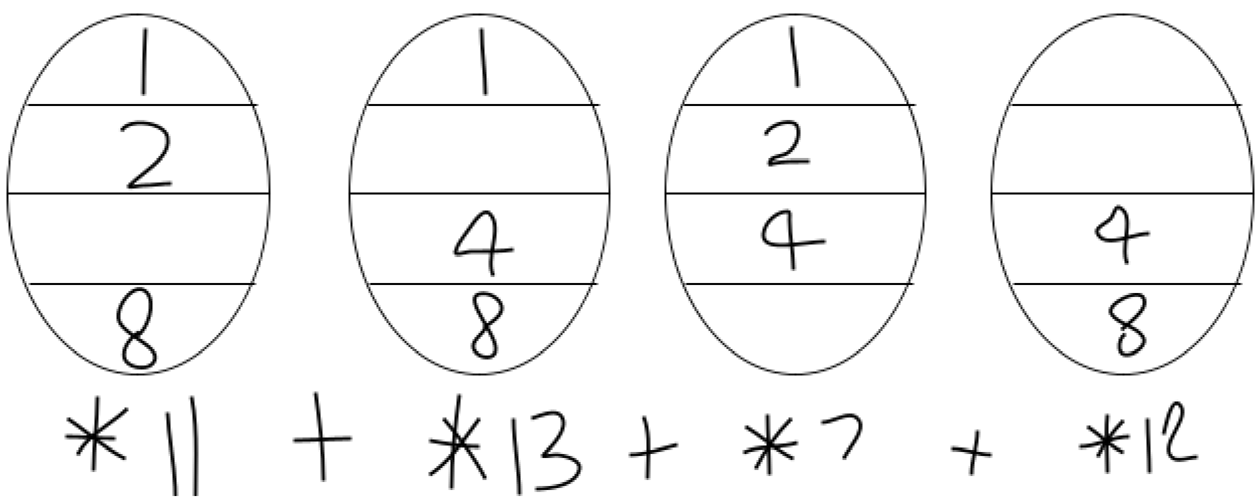
-Impartial Games-

- Definition: An impartial game is any combinatorial game where each player has the same possible moves (The only difference between players is who goes first)
- Correlation: Every position is type N or P. L/R positions do not exist because each player can make the same moves
- Correlation: A position type in an impartial game is type N if there exists any move to a type P position. A position is type P if there exists no move to a position of type P.

Nim

- Nimbers is an impartial game with L piles of stones (or other objects), $a_1 \ldots a_l$. A move is to pick one pile and remove as many stones as you want

(Nimbers represented in easter eggs with shown powers of 2)



Use *X to represent a pile of objects and the number of objects in the pile

- Def: Given a Nim position $*q_1 + \ldots + *q_k$, it is "balanced" if for every power of 2, the total number of sub-piles of that size is even.

BALANCE PROCEDURE

*0 is type P and even numbers of powers of 2.

Pick a stack with unbalanced power

Remove uneven so they would become even

Everything should be even

Proposition: Every balanced type P position and every unbalanced position is type N.

<u>Dfn. Nim-Sum</u>

- The nim-sum of the non-regular integer denoted $a_1 \oplus \ldots \oplus a_r$ is the non-negative integer b. Sit. 2^c appears in the binary exponent of b if and only if this term appears and odd # of times in the binary expansion of $a_1 \ldots, a_l$

Take the numbers 13, 19, and 10 and expand them to their powers of 2

$$13 \oplus 19 \oplus 10 = (\cancel{8} + 4 + \cancel{1}) \oplus (16 + \cancel{2} + \cancel{1}) \oplus (\cancel{8} + \cancel{2})$$

4 and 16 appear an odd number of times, so they are kept

$$= 4 + 16$$

$$= 20$$

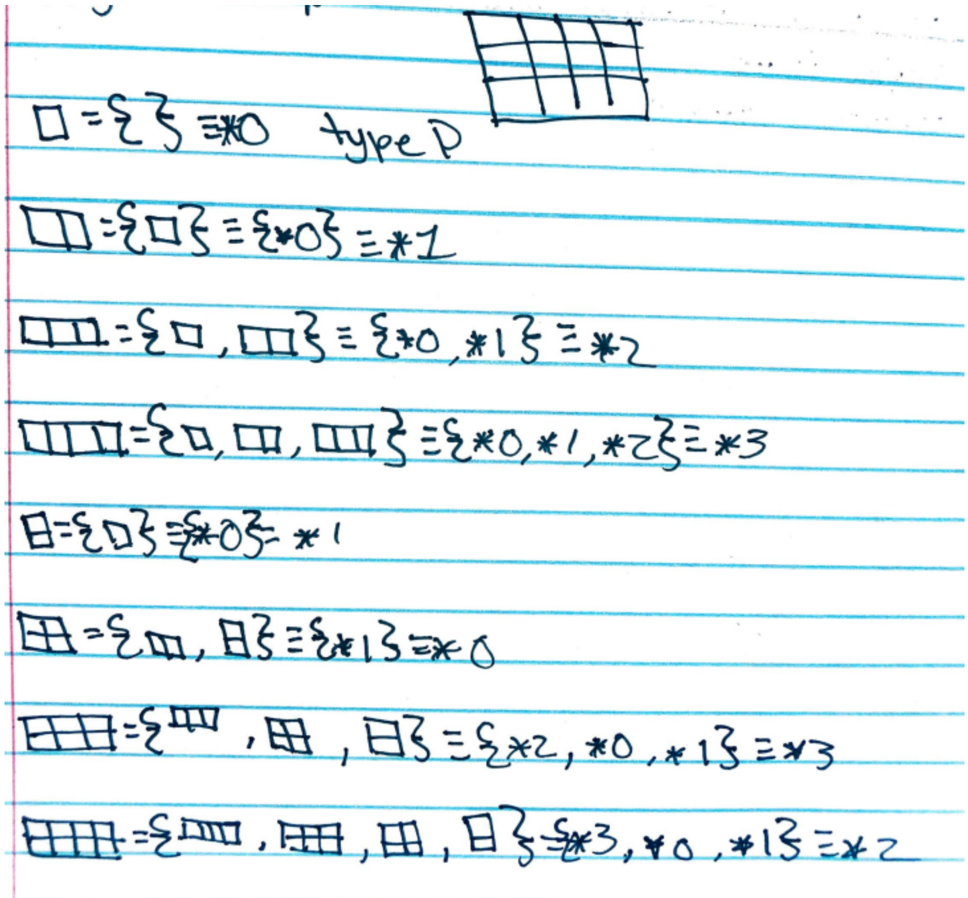
- Then, if $a_1, \ldots a_l$ are non-reg int and $b = a \oplus \ldots \oplus a_l$, then $*a_1 + *a_2 \ldots + *a_l = *b$
- Every Nim position is equal to a nimber
- Every impartial game corresponds to a nimber
- Nim *4 = {*0, *1, *2, *3}
- Then every position in an impartial game is equal to a number

- Def: For a set $S = \{a_1 \ldots a_n\}$ of non-neg integers, we define the minimal excluded value (MEX) of S to be the smallest non-negative integer b, which is not one of $a_1, \ldots a_n$
- A = {0,1,2,4,5,20} – then MEX (A) = 3
- The (MEX Principle) Let $A = \{a_1, a_2 \ldots a_k\}$ be a position in an impartial game

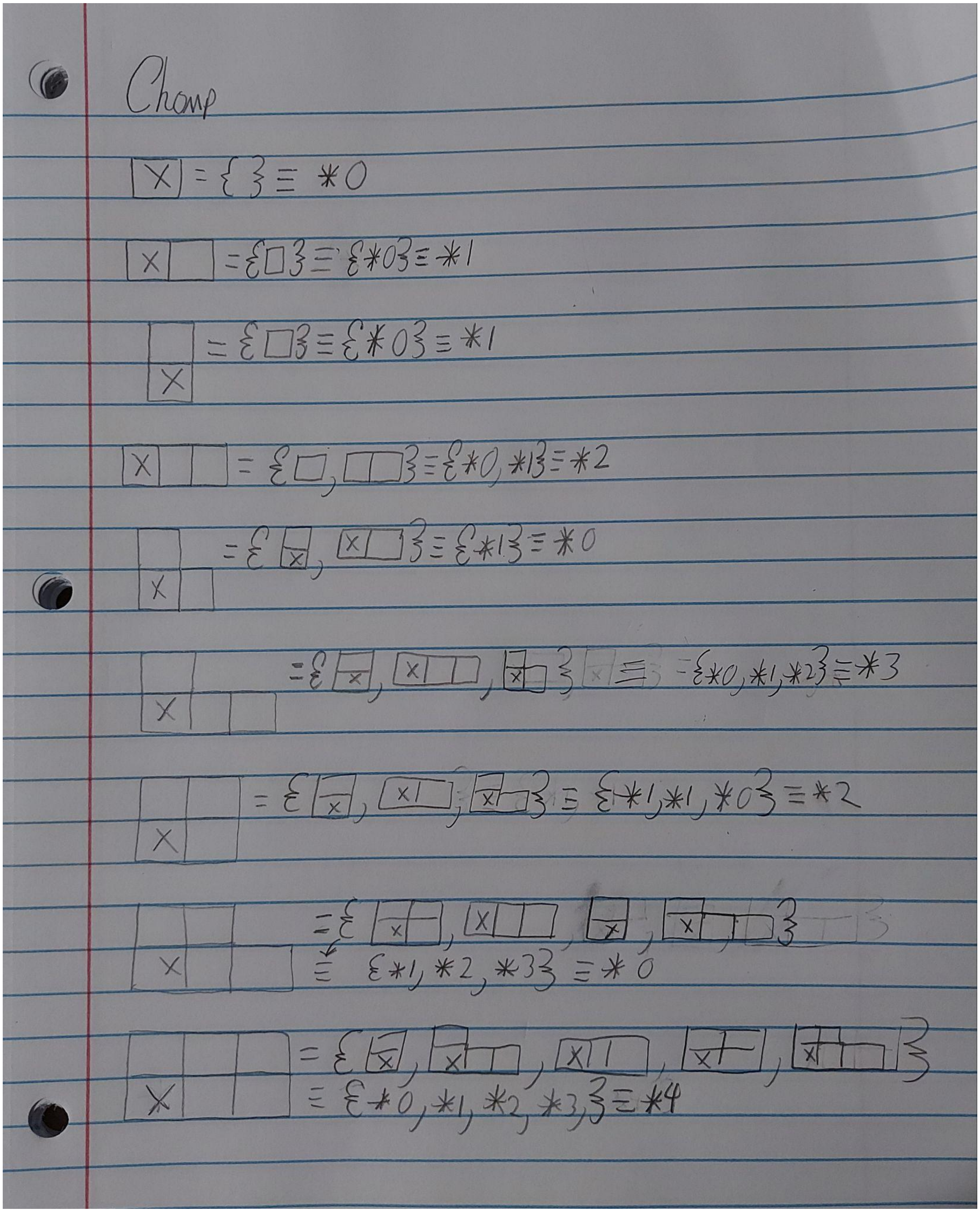
Suppose that a1 = *a1 for every 1<= I <= k

Then a = *b where b is the MEX of the set A

Apply to the game chop:



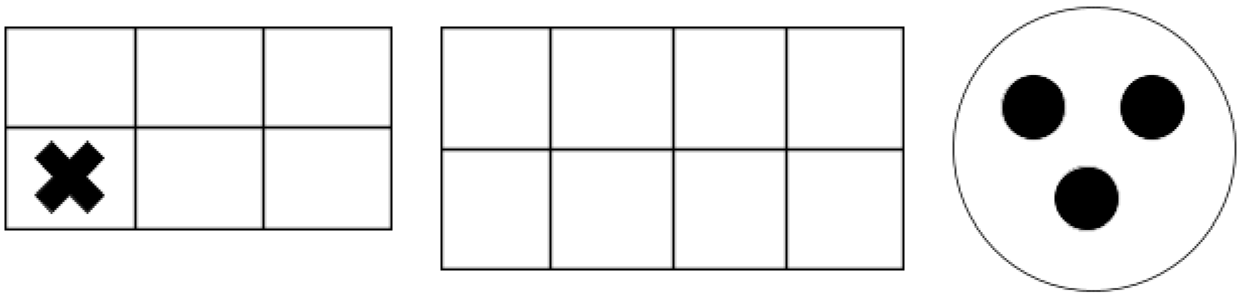
Apply to the game chomp:



Notes for 2/11

Finding equivalent sums

Each impartial game can be represented as a nimber. Even if the games are different, we can convert them to nimbers and add them together. This is principally the same as a game of nim, and can be played as such, using the balancing procedure to win.



Chop + Chomp + Nim

=

$*4 + *2 + *3 \equiv *(4 \oplus 2 \oplus 3) \equiv 5$

How to play - balancing

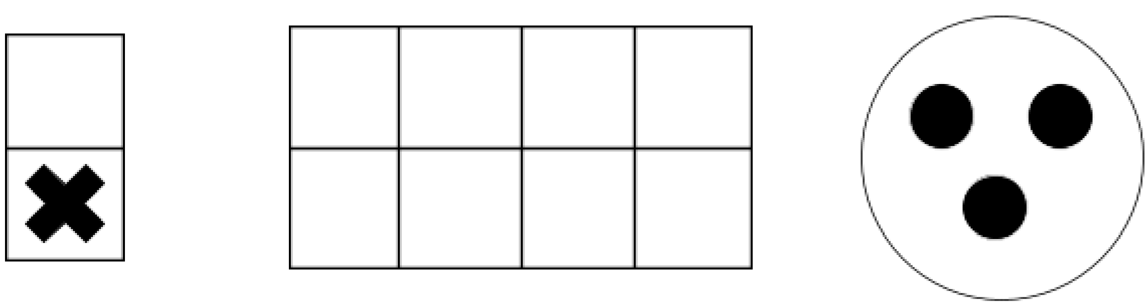
$*4 + *2 + *3 \equiv *4 + *2 + *(2+1)$

$2^2 + 2^1 + (2^1 + 2^0)$

2^2 and 2^0 are the uneven (odd) powers of two here, so we want to play the games in such a way that removes or changes the 2^2 and the 2^0, leaving us with an even number of powers of two, like this…

$2^0 + 2^1 + (2^1 + 2^0)$

So we need to turn the *4 game of chomp to a *1, like so



This game configuration is equivalent to *1 + *2 + *3

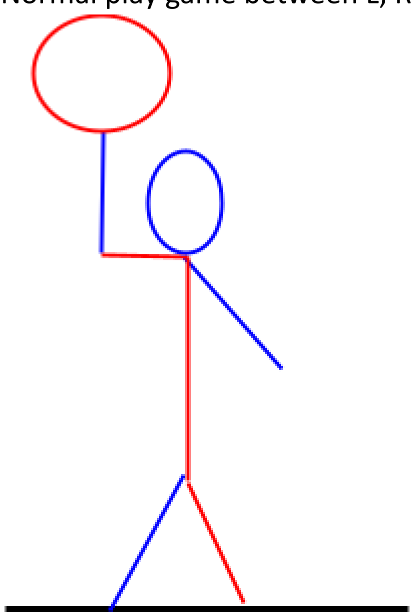
- **Pick up Bricks theorem: Let n = 3l + K where 0 ≤ K ≤ 2. Then a Pick up Bricks position of n bricks is equal to *K. Always *0, *1, *2**
- **Chop Theorem: For every m, n ≥ 1, an m x n position in chop is equivalent to *(m-1)+*(n - 1). 1 x 1 is terminal.**

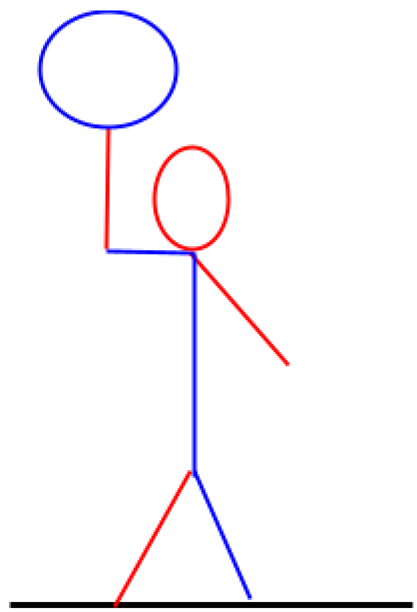
-Partizan Games-

- All normal impartial games are equivalent to a nimber
  - find a similar thing for partizan games

Hackenbush

- Normal play game between L, R.





∝ and -∝

- Defn: •0 is the Hackenbush position with no edges. •0 is type P
  - Prop: A HB position ∝ ≡ •0 iff ∝ is type P

The sum operator ∝ + •0 ≡ ∝ for all normal play games

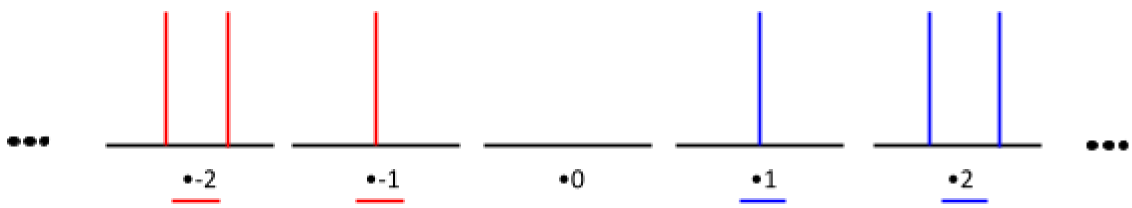
Negation: Reversing the colors, reverses the roles of the players, switching all colors is negation

Proposition: If ∝ and β are HB positions, then

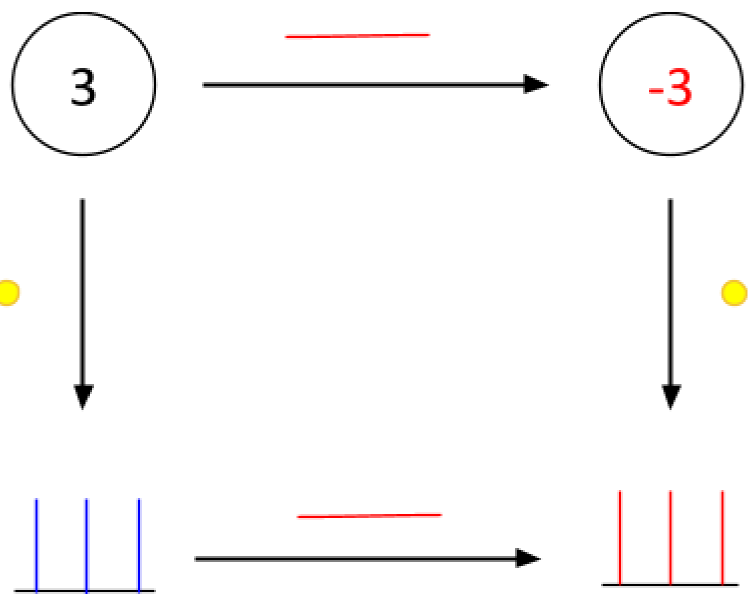
1. -(-∝) ≡ ∝ (reversing the reversal gives us the original)
2. ∝ + (-∝) ≡ •0
3. β + (-∝) ≡ •0 implies ∝ ≡ β

Integer positions

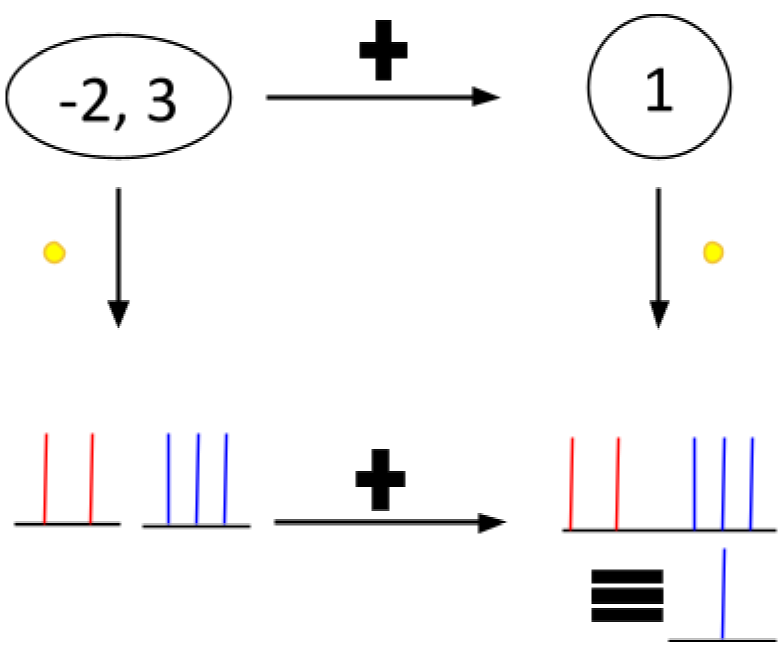
For every position integer n, define •n to be the HB position consisting of n isolalted blue edge



Visual of negation



Visual of addition



Theorem: For any integers

1. $(-\bullet n) \equiv \bullet(-n)$
2. $(\bullet m) + (\bullet n) \equiv \bullet(m + n)$

- $n > 0$ R has advantage
- $n < 0$ L has advantage

Brief Lab Notes

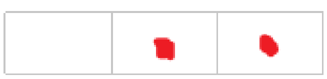
In the previous lab, we played a few games and tried to analyze type L, R, N, and P positions within them. As a quick reminder,

- Type L- L will always win no matter who goes first
- Type R - R will always win no matter who goes first
- Type N - The next player will always win
- Type P - The previous player will always win

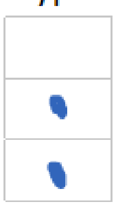
Here are a few game positions from the games we played in lab one

<u>Domineering</u> - Players play dominoes two squares in size. One player places horizontally, the other places vertically, first player who can't make a move loses

Type L Game configuration (L plays in dotted red squares to win)



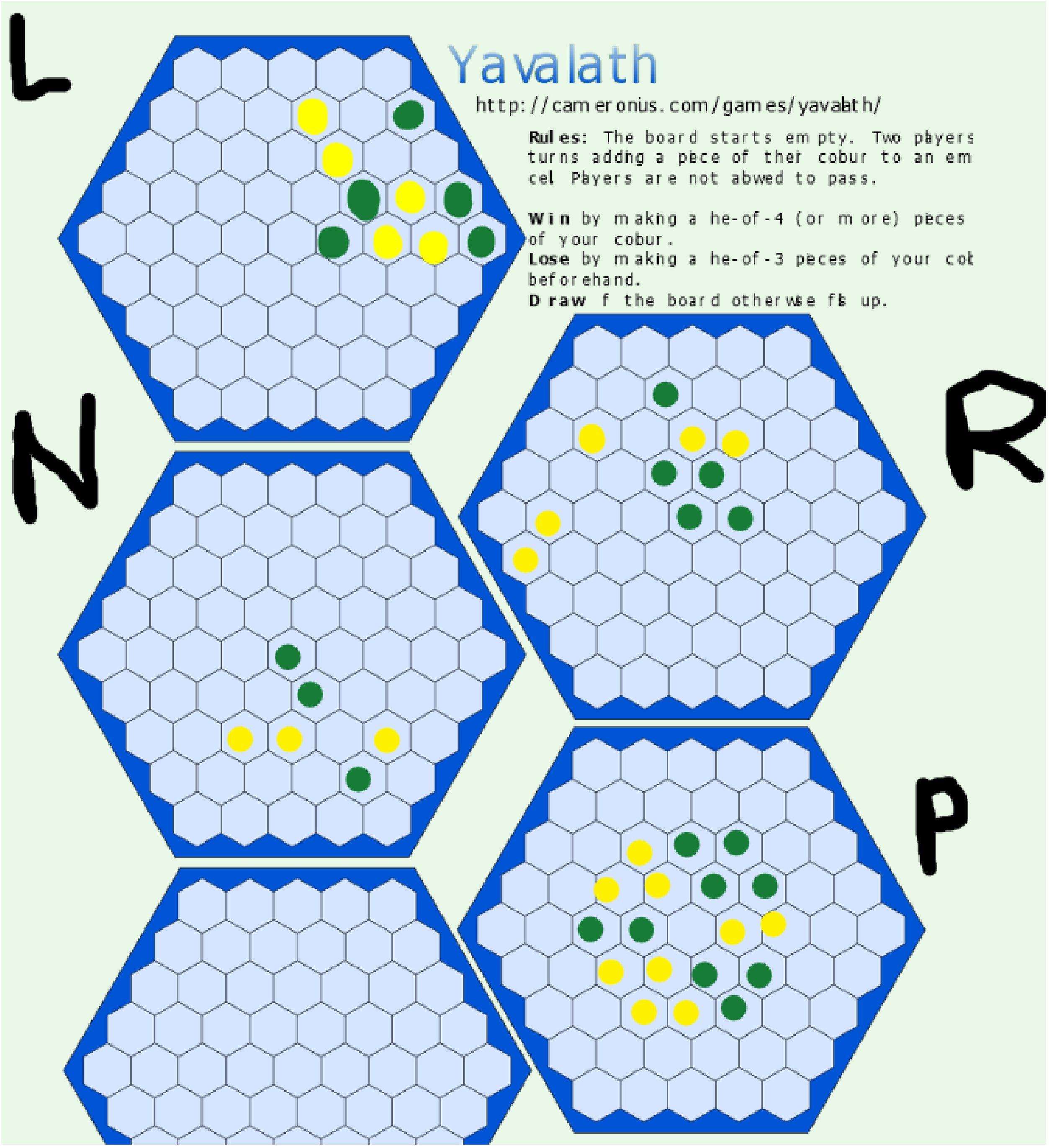
Type R Game configuration (R plays in dotted blue squares to win



Type N Game configuration (R wins with blue dots, L wins with red)

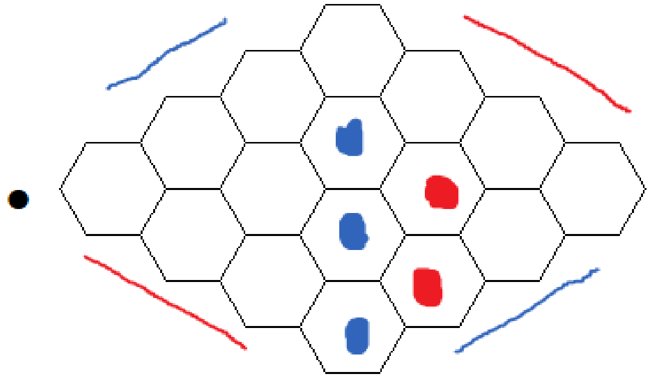
Type P Game configuration (Second player to go always wins)

<u>Yavalath</u> - Players take turns placing dots on a hex board. Make 4 in a row to win, 3 makes you lose the game

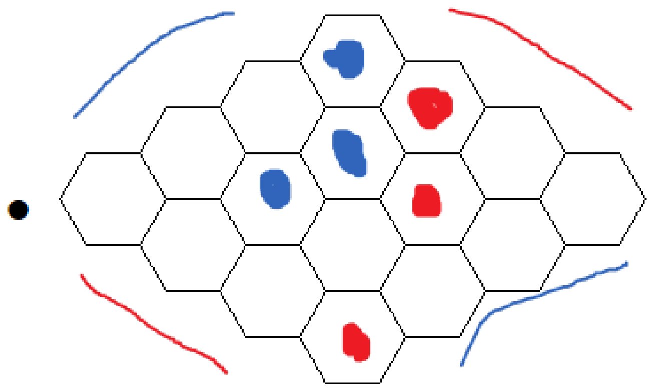


(For the type P board, assume we do not use the outer two layers)

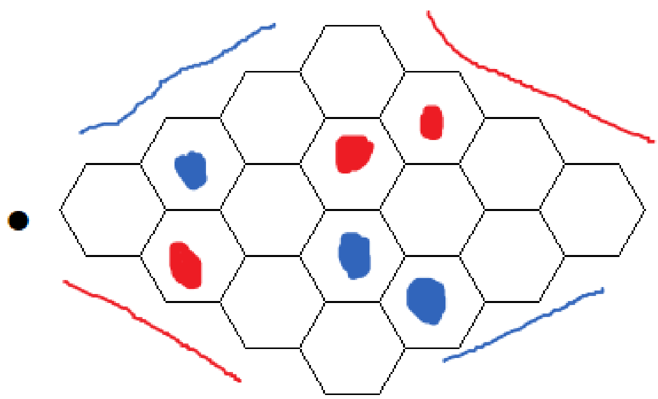
Hex - Players take turns placing dots on a hex board. The first to link their sides together with their colored pieces wins (we used smaller boards for ease of production)



Type L position



Type R position



Type N position

Note: We are not sure if a type P position exists here. A type P position would imply that somehow, the next player to go cannot win, and their move would be allowing the previous player to win. We can't run out of moves like in domineering, since draws are impossible in hex[1], and there's no way we can make ourselves lose like in yavalath. The way pieces are placed in hex means that you will always have a move to make until the game is over. If anyone knows a type P position, let us know. Thankyou and may God bless you.

[1] https://en.wikipedia.org/wiki/Hex_(board_game)#Determinacy